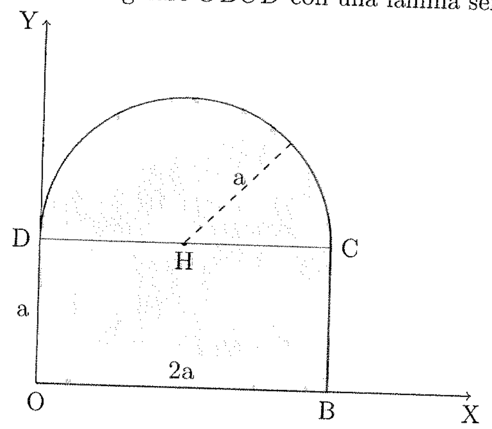
Problema 1 Si consideri la lamina omogenea di massa M schematizzata in figura e ottenuta saldando una lamina rettangolare $OBCD$ con una lamina semicircolare di raggio a e centro nel punto medio del lato DC.



Come indicato in figura le lunghezze dei lati della lamina rettangolare sono

$$\begin{cases} \overline{BO} = \overline{DC} = 2a \\ \overline{CB} = \overline{OD} = a \end{cases}$$

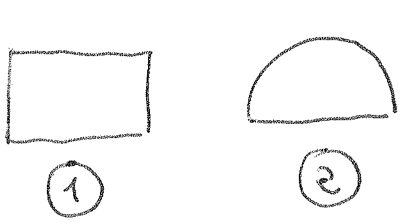
Si determinino:

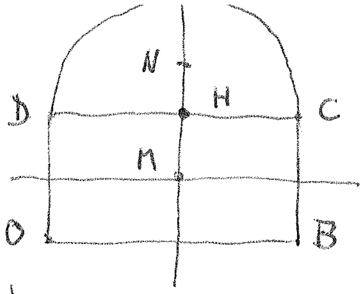
1. la *posizione del baricentro* della lamina;
2. la *matrice di inerzia* della lamina rispetto al polo O ed al sistema di assi $\{X, Y\}$.

Soluzione

1. $$\left(a \, , \, \frac{10+3\pi}{3(4+\pi)} a \right)$$

2. $$I_O = \frac{ma^2}{4+\pi} \begin{pmatrix} 4 + \frac{5\pi}{4} & -\left(\frac{10}{3} + \pi\right) \\ -\left(\frac{10}{3} + \pi\right) & \frac{16}{3} + \frac{5\pi}{4} \end{pmatrix}$$





$$\overline{HN} = \frac{4a}{3\pi}$$

$$A_{tot} = 2a \cdot a + \frac{\pi a^2}{2} = \frac{a^2}{2}(4+\pi) \qquad \rho = \frac{m}{A_{tot}} = \frac{2m}{a^2(4+\pi)}$$

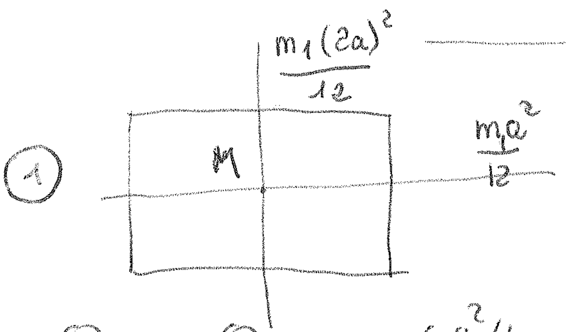
$$m_1 = 2a^2 \rho = \frac{4m}{4+\pi} \qquad m_2 = \frac{\pi a^2}{2}\rho = \frac{\pi m}{4+\pi}$$

1. $x_G = a$ perchè l'asse $x = a$ è asse di simmetria

$$y_G = \frac{m_1}{m} y_M + \frac{m_2}{m} y_N = \frac{4}{4+\pi}\frac{a}{2} + \frac{\pi}{4+\pi}\left(a + \frac{4a}{3\pi}\right)$$

$$= \frac{a}{4+\pi}\left(2 + \pi + \frac{4}{3}\right) = \frac{10+3\pi}{3(4+\pi)}\,a$$

2.

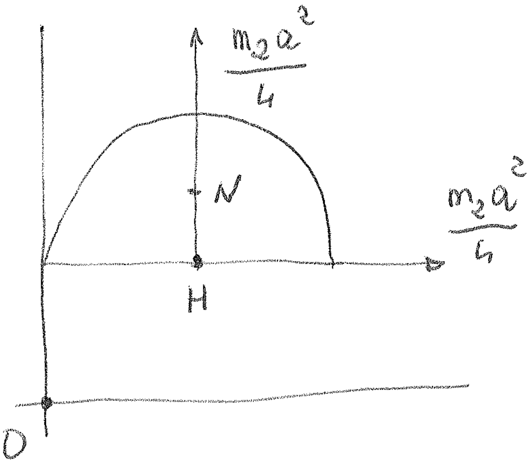


$$I_O^{(1)} = I_M^{(1)} + m_1 \begin{pmatrix} a^2/4 & -a^2/12 \\ -a^2/2 & a^2 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{m_1 a^2}{12} & 0 \\ 0 & \frac{m_1 a^2}{3} \end{pmatrix} + m_1 \begin{pmatrix} a^2/4 & -a^2/12 \\ -a^2/12 & a^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{m_1 a^2}{3} & -\frac{m_1 a^2}{2} \\ -\frac{m_1 a^2}{2} & \frac{4 m_1 a^2}{3} \end{pmatrix} = \frac{m a^2}{4+\pi} \begin{pmatrix} \frac{4}{3} & -2 \\ -2 & \frac{16}{3} \end{pmatrix}$$

(2)



$$\overline{HN} = \frac{4a}{3\pi}$$

$$I_O^{(2)} = I_H^{(2)} - m_2 \begin{pmatrix} \overline{HN}^2 & 0 \\ 0 & 0 \end{pmatrix} + m_2 \begin{pmatrix} (\overline{HN}+a)^2 & -(HN+a)a \\ -(HN+a)a & a^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{m_2 a^2}{4} & 0 \\ 0 & \frac{m_2 a^2}{4} \end{pmatrix} + m_2 \begin{pmatrix} -\overline{HN}^2 + \overline{HN}^2 + a^2 + 2\frac{4a}{3\pi}a & -\left(\frac{4}{3\pi}+1\right)a^2 \\ -\left(\frac{4}{3\pi}+1\right)a^2 & a^2 \end{pmatrix}$$

$$= m_2 a^2 \begin{pmatrix} \frac{1}{4} + 1 + \frac{8}{3\pi} & -\left(\frac{4}{3\pi}+1\right) \\ -\left(\frac{4}{3\pi}+1\right) & \frac{1}{4}+1 \end{pmatrix}$$

$$= \frac{ma^2}{4+\pi} \begin{pmatrix} \frac{5\pi}{4} + \frac{8}{3} & -\left(\frac{4}{3}+\pi\right) \\ -\left(\frac{4}{3}+\pi\right) & \frac{5\pi}{4} \end{pmatrix}$$

$$I_O = I_O^{(1)} + I_O^{(2)}$$

$$= \frac{ma^2}{4+\pi} \begin{pmatrix} \frac{4}{3} + \frac{5\pi}{4} + \frac{8}{3} & -\left(2+\frac{4}{3}+\pi\right) \\ -\left(2+\frac{4}{3}+\pi\right) & \frac{16}{3} + \frac{5\pi}{4} \end{pmatrix}$$

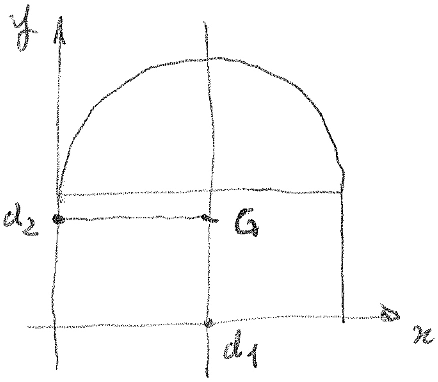
ossie

$$I_O = \frac{ma^2}{4+\pi}\begin{pmatrix} 4+\frac{5\pi}{4} & -\left(\frac{10}{3}+\pi\right) \\ -\left(\frac{10}{3}+\pi\right) & \frac{16}{3}+\frac{5\pi}{4} \end{pmatrix}$$

Altro modo per il calcolo di I^O_{12}

Dalla legge di trasposizione della matrice d'inerzia si ha

$$I^O_{12} = I^G_{12} - m d_1 d_2 = -m d_1 d_2$$



Dalla figura si ha

$$d_1 = x_G \,, \quad d_2 = y_G$$

da cui

$$I^O_{12} = -m x_G y_G =$$

$$= -m \frac{10+3\pi}{3(4+\pi)} a^2 = -\frac{ma^2}{4+\pi}\left(\frac{10}{3}+\pi\right)$$

come già ottenuto