J Optim Theory Appl (2011) 150:516–529
DOI 10.1007/s10957-011-9851-1

Vector Optimization Problems via Improvement Sets

M. Chicco · F. Mignanego · L. Pusillo · S. Tijs

Published online: 27 April 2011

Abstract Motivated by applications to the real world, various optimality criteria (also approximate ones) are developed for situations in vector optimization. We propose a new type of solution based on upper comprehensive sets and we discuss the existence of optimal points in multicriteria situations.

Keywords Vector optimization · Pareto equilibria · Approximate solutions · Improvement sets

Communicated by Panos M. Pardalos.

This research was partially supported by MUR (Ministero Università Ricerca—Italia) via a contract with S. Tijs. We thank Rodica Branzei for her helpful comments on a previous version of the paper. The valuable comments of the referees are gratefully acknowledged.

M. Chicco
Diptem, Sezione Metodi e Modelli Matematici, Facoltà di Ingegneria, University of Genoa, P. le Kennedy, pad. D, 16129 Genoa, Italy
e-mail: chicco@diptem.unige.it

F. Mignanego
Department of Mathematical Disciplines, Mathematical Finance and Econometrics, Catholic University Milan, Milan, Italy
e-mail: fausto.mignanego@unicatt.it

L. Pusillo (✉)
Dima, Department of Mathematics, University of Genoa, via Dodecaneso 35, 16146 Genoa, Italy
e-mail: pusillo@dima.unige.it

S. Tijs
CentER and Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands
e-mail: s.h.tijs@uvt.nl

1 Introduction

Decisions fill our life, but often these involve several objectives sometimes non comparable, so real problems have to be solved optimally according to many criteria. Consider, for example, if it was decided to build a new major international airport, how would a government select a site? It would need to be far from mountains, city centers, pre-existing airports and air corridors. As the government has to minimize the problems for the traveller so it would need to be near the country's major city. Further to this, the airport should be sited as far away as possible from cities, to maximize safety and minimize problems for the population. The government would wish to minimize the construction costs and maximize capacity. It would also wish to increase the country's international prestige. As you see there are many objectives to "maximize" and often ones not comparable (see [1] for many interesting decision examples).

In general, there is not a solution, which pleases every decision maker, under each criteria considered, so the theory of vector optimization arised.

We should remind you that when we speak about a multiobjective problem with objective functions to "maximize" (resp. to "minimize") we have a feasible set and the objective functions f_i which are also called *criteria*.

The optimization problem (called also multicriteria problem) consists of "maximize" (resp. "minimize") all these functions. "Maximize" or "minimize" in a sense to define. The criterion space is the space from which the criterion values are taken.

We will use indifferently the words multicriteria or multiobjective or vector optimization problem.

The first papers in this research area were published by Edgeworth and Pareto: for an historical development of Vector Optimization, see [2], about vector optimization problems see [3, 4] and for economic applications [5]. After the fundamental work of Pareto, many optimal criteria, also approximate ones, were proposed in literature ([7–9]).

There are many books which consider this fascinating theory with a deep mathematical approach (see for example [10–17] and references therein).

The notion of approximate solution in Vector Optimization is far from being uniquely defined, so it is not easy to choose a proper concept of maximizing (minimizing) sequences as in well-posedness properties, see [6, 18–20].

Approximate solutions play an important role, when there are no exact solutions. In optimization problems, the models are representations of the real world and many of these are solved using iterative algorithms which give us approximate solutions.

In this paper, we introduce in a new way (approximate) multicriteria solutions based on special sets E in $\mathbb{R}^n$, called improvement sets.

The improvement set has two properties: the exclusion property and the comprehensive property.

Intuitively the elements of the improvement set can be seen as essential improvements of $\mathbf{0}$. It expresses the view of the decision maker on what is substantially better.

Through the improvement sets we give a new concept of optimal points. For example to find the weak Pareto points of a subset of an Euclidean space, we consider as improvement set that one with all the coordinates strictly positive.

This new notion of optimal point generalizes the concept of optimal points and approximate optimal points in scalar optimization and the concept of Pareto equilibria and approximate equilibria. In fact by this definition we keep into account contemporarily the idea of approximate and exact solution.

The work is structured as follows: in Sect. 2 we give some preliminary results and we introduce some properties about improvement sets; in Sect. 3 we give the new definition of optimal point, we prove that this generalizes other notions known in literature, we give some topological properties about this new class and we prove some relations between an improvement set and the epigraph of a decreasing function; in Sect. 4 we prove the existence results of these optimal points and many examples clarify the presentation.

Finally in Sect. 5 we present the conclusion that summarizes the work and we give some ideas of open problems.

2 Definitions and Preliminary Results

We write $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\mathbf{x}_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$, and $E_{-i} = \{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1} : \exists\, (x_1, \dots, x_i, \dots, x_n) \in E \subset \mathbb{R}^n\}$.

Let $\mathbf{e}_1 = (1, 0, \dots, 0), \mathbf{e}_2 = (0, 1, \dots, 0), \dots,$ in general $\mathbf{e}_i$ is the vector with component i equal 1 and the others 0 so that for every $\mathbf{x} \in \mathbb{R}^n$ with components $x_1, x_2, \dots, x_n$ we have $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i$.

Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we consider the following inequalities on $\mathbb{R}^n$:

$\mathbf{x} \geqq \mathbf{y} \Leftrightarrow x_i \geq y_i \ \forall i = 1, \dots, n$;
$\mathbf{x} \geq \mathbf{y} \Leftrightarrow \mathbf{x} \geqq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$;
$\mathbf{x} > \mathbf{y} \Leftrightarrow x_i > y_i \ \forall i = 1, \dots, n$
(analogous definitions for $\leqq$, $\leq$, $<$).

For $A \in \mathbb{R}^n$ we denote:

$int(A)$ is the interior of A;
$ri(A)$ is the relative interior of A;
$bd(A)$ is the boundary of A;
$cl(A) = int(A) \cup bd(A)$ is the closure of A.

We say that $V \subset \mathbb{R}^n$ is *upper bounded* (u.b. for short) iff there exists $\mathbf{b} \in \mathbb{R}^n$ such that $\mathbf{x} \leqq \mathbf{b}\ \forall \mathbf{x} \in V$.

By $\mathbb{R}^n_+$ we mean the points in $\mathbb{R}^n$ with all coordinates positive or null, by $\mathbb{R}^n_{++}$ we mean the points in $\mathbb{R}^n$ with all coordinates strictly positive (analogously for $\mathbb{R}^n_-$ and $\mathbb{R}^n_{--}$).

We point out that a feasible point in $\mathbb{R}^n$ is *strongly Pareto-optimal* (we indicate the set of strongly Pareto points as *sPE*) iff there is no other feasible point which is larger in at least one coordinate and not smaller in all the other coordinates.

A feasible point in $\mathbb{R}^n$ is *weakly Pareto-optimal* (*wPE* for short), iff there is no other feasible point which is larger in each coordinate.

We briefly indicate $PE(A)$ to mean all the Pareto points of the set A.

In the following we will write $\langle \mathbf{a}, \mathbf{b} \rangle$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, to indicate the internal (scalar) product of two vectors.

Let $\Delta = \{\mathbf{p} \in \mathbb{R}^n_+ : \sum_{i=1}^n p_i = 1\}$ and $\Delta^0 = ri(\Delta)$.
It is easy to prove the following proposition:

Proposition 2.1 *Let* $\mathbf{p} \in \Delta^0$ *and let* $A \subset \mathbb{R}^n$, *then*

$$\operatorname{argmax}\big\{\langle \mathbf{p}.\mathbf{a}\rangle : \mathbf{a} \in A\big\} = \Big\{\tilde{\mathbf{a}} \in A : \langle \mathbf{p}.\tilde{\mathbf{a}}\rangle = \max_A \langle \mathbf{p}.\mathbf{a}\rangle\Big\} \subset PE(A).$$

Note that $\operatorname{argmax}\{\langle \mathbf{p}.\mathbf{a}\rangle : \mathbf{a} \in A\}$ may be empty, for example if we consider $A = \mathbb{R}^n_{--}$.

Definition 2.1 Let $f : X \to \mathbb{R}$ with $X \subset \mathbb{R}^n$; f is said to be decreasing with respect to all of its variables iff $f(\mathbf{x}) \geq f(\mathbf{x} + h\mathbf{e}_i)$ for all $\mathbf{x} \in X$, for all $h > 0$ and for all $i = 1, 2, \ldots, n$ such that $\mathbf{x} + h\mathbf{e}_i \in X$.

It follows that $f : X \to \mathbb{R}$ is decreasing w.r.t. all its variables if and only if, given any $\mathbf{x}, \mathbf{y} \in X$ with $\mathbf{x} \geqq \mathbf{y}$, we have $f(\mathbf{x}) \leq f(\mathbf{y})$.

Definition 2.2 Let $f : Y \to \mathbb{R}$, $Y \subset \mathbb{R}^{n-1}$, we call *epigraph* of the function f (w.r.t. the nth coordinate) and we will write *epi f* the set:

$$epi f = \big\{\mathbf{x} \in \mathbb{R}^n : x_n \geq f(x_1, x_2, \ldots, x_{n-1})\ \forall (x_1, x_2, \ldots, x_{n-1}) \in Y\big\}.$$

Definition 2.3 Let $A \subset \mathbb{R}^n$. We define the upper comprehensive set of A, and we shall write:

$$u\text{-}compr(A) = \big\{\mathbf{x} \in \mathbb{R}^n :\ \text{there is } \mathbf{a} \in A \text{ s.t. } \mathbf{a} \leqq \mathbf{x}\big\}.$$

It is easy to prove:

Proposition 2.2 $u\text{-}compr(A) = \bigcup_{a \in A}(a + \mathbb{R}^n_+)$.

Definition 2.4 A subset E of $\mathbb{R}^n$ is called "upper comprehensive set" iff $u\text{-}compr(E) = E$.

Definition 2.5 Let $E \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ be an upper comprehensive set. We shall call E an improvement set of $\mathbb{R}^n$ and we will write the family of the improvement sets in $\mathbb{R}^n$: $\Im^n$.

Remark 2.1 From the above definitions, it follows that $E \subset \mathbb{R}^n$ is an improvement set iff it has the two following properties:

(i) $\mathbf{0} \notin E$;
(ii) if $\mathbf{x} \in E$ and $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} \geqq \mathbf{0}$ then $\mathbf{x} + \mathbf{y} \in E$.

Remark 2.2

(i) If $n = 1$ the elements of $\Im^1$ are: $\emptyset$, $]0, +\infty[$, $]\epsilon, +\infty[$, $[\epsilon, +\infty[$ with $\epsilon > 0$.

(ii) In [21] the improvement set is

$$E = \mathbb{R}^2_+ \setminus [0,\epsilon] \times [0,\epsilon].$$

(iii) In [20] the improvement set is

$$E = \left\{\mathbf{x} \in \mathbb{R}^n : x_i \geqq \epsilon_i, \epsilon_i \in \mathbb{R}_{++},\ i = 1,\dots,n\right\}.$$

In the following remark we see some properties of the improvement sets.

Remark 2.3 We recall that a set $B \in \mathbb{R}^n$ is *solid* iff *int* $B \neq \emptyset$;
B is *proper* iff $\mathbb{R}^n \neq B \neq \emptyset$;
B is *pointed* iff $B \cap (-B) \subset \{\mathbf{0}\}$.
So $E \in \Im^n$ may be solid, proper but not a pointed set.
We recall that a set $B \in \mathbb{R}^n$ is called *radiant*, iff $\alpha\mathbf{d} \in B$ $\forall \mathbf{d} \in B$, $\forall \alpha > 1$.
An improvement set might not be *radiant* as $\{(x,y) \in \mathbb{R}^2 : y > \lg_{1/2}(x+1)\}$, and a radiant set might not be an improvement set such as $\mathbb{R}^2 \setminus [0,\epsilon] \times [0,\epsilon]$, but there are sets which are radiant and improvement sets such as $\mathbb{R}^2_{++}$.
We recall that $K \subset \mathbb{R}^n$ is called *cone* iff

$$\alpha\mathbf{d} \in K, \quad \forall \mathbf{d} \in K,\ \forall \alpha \in \mathbb{R},\ \alpha > 0$$

and a set F is called *free disposal* iff for all cones $K \subset \mathbb{R}^n$, K closed, convex and pointed it turns out

$$F + K = F.$$

So we can see that an improvement set is not a free disposal set: it is sufficient to consider cones whose magnitude is larger than $\pi/2$.

3 Improvement Sets and E-Optimal Points

In this section, we give the new notion of optimal points and some topological properties about them.

We start with a proposition to show some properties of the family $\Im^n$, which will be useful later.

Proposition 3.1 *The following relations are valid*:

(i) $\Im^n$ *is a lattice*, *that is*: *if* $E_1, E_2 \in \Im^n$, *then*

$$E_1 \cap E_2 \in \Im^n,\ E_1 \cup E_2 \in \Im^n.$$

(ii) *The largest improvement set is* $E_+ = \mathbb{R}^n \setminus \mathbb{R}^n_-$.
(iii) *The smallest is* $E_- = \emptyset$.

Proof We only prove point (ii) leaving the other ones to the readers.

Let us prove that E_+ is an improvement set; in fact $\mathbf{0} \notin E_+$, and if $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} \geqq \mathbf{0}$ and $\mathbf{x} \in E_+$, then $\mathbf{x}+\mathbf{y}$ has at least one strictly positive coordinate and we add a point with each coordinate larger or equal to zero, so $\mathbf{x}+\mathbf{y} \in E_+$.

Conversely, if $E \subset \mathbb{R}^n$, E is an improvement set, then $E \subset E_+$. In fact let us suppose there exists $\overline{\mathbf{x}} \in E \setminus E_+$, so $\overline{x_i} \leq 0$ and $-\overline{x_i} \geq 0$ $\forall i = 1, \dots, n$ and $\mathbf{0} = \overline{\mathbf{x}} + (-\overline{\mathbf{x}}) \in E$, which is a contradiction. □

Now we give a new definition of optimal point:

Definition 3.1 Given an improvement set E and a set A of alternatives for a decision maker (or a team), we say that $\mathbf{a} \in A$ is E-optimal iff $(\mathbf{a} + E) \cap A = \emptyset$ and denote this by using $\mathbf{a} \in O^E(A)$.

Remark 3.1 In other words: if $A \subset \mathbb{R}^n$ and $E \in \Im^n$, we say that $\mathbf{a} \in O^E(A)$ iff $\mathbf{a} \in A$ and $\mathbf{a} + \mathbf{e} \notin A$ $\forall \mathbf{e} \in E$.

Remark 3.2

(i) The previous definition is a generalization of that used in scalar optimization, in fact we recall that, if $n = 1$, then $\hat{a} \in A$ is an optimal point if: $[\hat{a}, +\infty[\cap A = \{\hat{a}\}$ if and only if $(\hat{a} + [0, +\infty[) \cap A = \{\hat{a}\}$ if and only if $(\hat{a} +]0, +\infty[) \cap A = \emptyset$. So here $E =]0, +\infty[$ is the improvement set.

(ii) If $E = \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, then $O^E(A)$ equals the set $sP(A)$ of the strong Pareto points of A.

(iii) If $E = \mathbb{R}^n_{++}$, then $O^E(A)$ equals the set $wP(A)$ of weak Pareto points of A.

To understand this new definition of optimal points, we present the following examples:

Example 3.1

(a) If $E = \mathbb{R}^2_+ \setminus [0, \epsilon] \times [0, \epsilon]$ and $A =]-\infty, 2] \times]-\infty, 2]$ it turns out $O^E(A) =]2-\epsilon, 2] \times]2-\epsilon, 2]$;
(b) if $E =]1, +\infty[$ and $A =]0, 10[$ then $O^E(A) = [9, 10[$;
(c) if $E =]0, +\infty[$ and $A =]0, 10[$ then $O^E(A) = \emptyset$;
(d) if $E =]0, +\infty[$ and $A =]0, 10]$ then $O^E(A) = \{10\}$;
(e) if $E = E_+$ and $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ then $O^E(A) = \emptyset$;
(f) if $E = \{(x, y) \in \mathbb{R}^2 : x + y \geq 0\} \setminus \{(0, 0)\}$ and $A = \{(x, y) \in \mathbb{R}^2 : x + y \leq 10\}$ then $O^E(A) = \emptyset$.

Example 3.2 If $E = \{(x, y) \in \mathbb{R}^2 : x + y > \delta > 0\}$, $A = [0, 2] \times [0, 2]$ then $O^E(A) = \{(x, y) \in \mathbb{R}^2 : x + y \geq 4 - \delta\}$. See Fig. 1. Let us prove it.

"$\subset$" If $(x, y) \in O^E(A)$ then $x + y > 4 - \delta$ in fact if, by absurdum $x + y \leq 4 - \delta$ then we choose $a = 2 - x$, $b = 2 - y$ then $a + b = 4 - x - y \geq \delta$ so $(2 - x, 2 - y) \in E$ and $(x, y) + (2 - x, 2 - y) = (2, 2) \in A$: a contradiction.

"$\supset$" Let $(x, y) \in \mathbb{R}^2$ s.t. $x + y > 4 - \delta$, consider $(a, b) \in E$, that is: $a + b \geq \delta$, then $(x, y) + (a, b) = (x + a, y + b)$ and $x + a + y + b > 4 - \delta + \delta = 4$, so $(x + a, y + b) \notin A$.

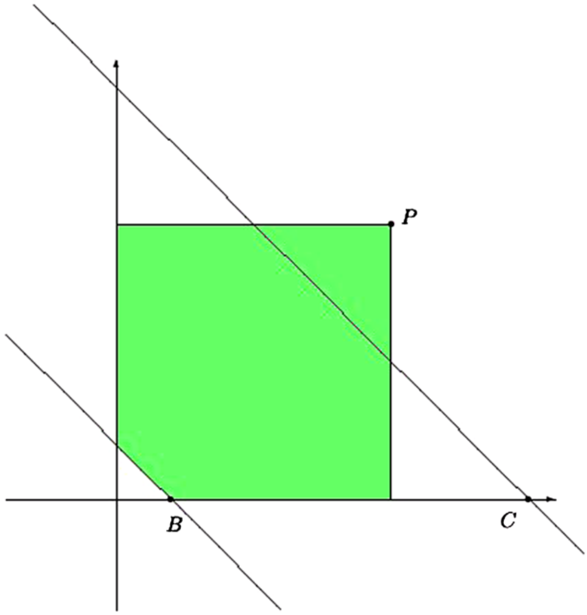

Fig. 1 $P=(2,2)$, $B=(4-\delta,0)$, $C=(\delta,0)$. Obviously the figure is made with δ fixed (and $\delta>2$), the *colored part* corresponds to $O^E(A)$

Proposition 3.2 *Let E_1, E_2 be improvement sets in $\mathbb{R}^n$. Then the following relations are valid*:

(i) *If $E_1 \supseteqq E_2$ then $O^{E_1}(A) \subseteqq O^{E_2}(A)$ for all $A \subset \mathbb{R}^n$.*

Intuitively if the decision maker is more flexible, *he* (*she*) *has more optimal points*.

(ii) $O^{\emptyset}(A)=\emptyset$;

(iii) $O^{E_+}(A)=\begin{cases}\max A, & \text{if } \max A \textit{ exists},\\ \emptyset & \textit{otherwise};\end{cases}$

(iv) $O^E(A)\subset A$;

(v) $O^{E_1\cup E_2}(A)= O^{E_1}(A)\cap O^{E_2}(A)$;

(vi) $O^{E_1}(A)\cup O^{E_2}(A)\subset O^{E_1\cap E_2}(A)$;

(vii) $\bigcap_{E':E'\subset E} O^{E'}(A)=O^E(A)$;

(viii) $\bigcap_{\epsilon>0} O^{]\epsilon,+\infty[}(A)=O^{]0,+\infty[}(A)$;

(ix) $\bigcap_{E:E\subset\mathbb{R}^n_+\setminus\{0\}} O^E(A)=sPE(A)$ (*analogously for wPE*).

Proof (i) $\mathbf{x}\in O^{E_1}(A)$ iff $\mathbf{x}\in A$ and $(\mathbf{x}+\mathbf{e}_1)\cap A=\emptyset$ $\forall\mathbf{e}_1\in E_1$ and in particular $\forall\mathbf{e}_2\in E_2$ being $E_2\subset E_1$.

(ii) and (iv) are obvious from the definition.

(iii) Let $\overline{\mathbf{x}}=\max A\in A$ (it means that $\overline{x_i}\geq x_i,\ i=1,\dots,n,\ \forall\mathbf{x}\in A$). Let us consider $\mathbf{y}\in E_+$, by definition there is at least $\bar{i}$ s.t. $y_{\bar{i}}$ is positive, so $\overline{\mathbf{x}}+\mathbf{y}\notin A$, then $\overline{\mathbf{x}}\in O^{E_+}(A)$.

Conversely let $\overline{\mathbf{x}}\in O^{E_+}(A)$, then $\overline{\mathbf{x}}+\mathbf{y}\notin A$ $\forall\mathbf{y}\in E_+$, so on for each point in $\mathbb{R}^n$ with a component larger of the corresponding component of $\overline{\mathbf{x}}$. Then each point in A has components $x_i\leq\overline{x_i}$, $i=1,\dots,n$.

If there is not max A then the thesis is obvious.

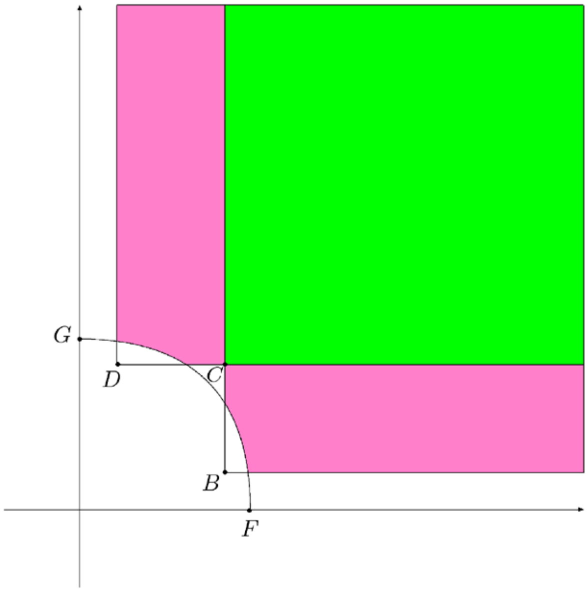

Fig. 2 To draw this picture we have fixed $\lambda = 1/2$. $G = (0, 1)$, $F = (1, 0)$, $D = (\frac{\sqrt{2}-1}{2}, \frac{2\sqrt{2}+1}{4})$, $B = (\frac{2\sqrt{2}+1}{4}, \frac{\sqrt{2}-1}{2})$, $C = (\frac{2\sqrt{2}+1}{4}, \frac{2\sqrt{2}+1}{4})$, $E_1 = \{(x, y) \in \mathbb{R}^2 : x \geq \frac{\sqrt{2}-1}{2}, y \geq \frac{2\sqrt{2}+1}{4}\}$, $E_2 = \{(x, y) \in \mathbb{R}^2 : x \geq \frac{2\sqrt{2}+1}{4}, y \geq \frac{\sqrt{2}-1}{2}\}$

(v) "$\subset$" Since $E_1 \cup E_2 \supset E_1$, $E_1 \cup E_2 \supset E_2$, we have

$$O^{E_1 \cup E_2}(A) \subset O^{E_1}(A) \text{ and } O^{E_1 \cup E_2}(A) \subset O^{E_2}(A).$$

"$\supset$" Take $\mathbf{x} \in O^{E_1}(A) \cap O^{E_2}(A)$. Then $\forall \mathbf{b}_1 \in E_1, \mathbf{x} + \mathbf{b}_1 \notin A$, $\forall\, \mathbf{b}_2 \in E_2, \mathbf{x} + \mathbf{b}_2 \notin A$, so $\mathbf{x} + \mathbf{b} \notin A$ for all $\mathbf{b} \in E_1 \cup E_2$. Hence $\mathbf{x} \in O^{E_1 \cup E_2}(A)$.

(vi) Take $\mathbf{a} \in O^{E_1}(A) \cup O^{E_2}(A)$, then $\mathbf{a} + \mathbf{b}_1 \notin A\ \forall \mathbf{b}_1 \in E_1$ or $\mathbf{a} + \mathbf{b}_2 \notin A\ \forall \mathbf{b}_2 \in E_2$. This implies: $\mathbf{a} + \mathbf{b} \notin A\ \forall \mathbf{b} \in E_1 \cap E_2$ so $\mathbf{a} \in O^{E_1 \cap E_2}(A)$.

(vii) "$\supset$" $O^{E'}(A) \supset O^{E}(A)\ \forall E' \subset E$ for (i) then $\bigcap_{E' : E' \subset E} O^{E'}(A) \supset O^{E}(A)$.

"$\subset$" Let $\mathbf{x} \in \bigcap_{E' : E' \subset E} O^{E'}(A)$ then $(\mathbf{x} + \mathbf{e}') \notin A\ \forall \mathbf{e}' \in E' \subset E$, if we choose $\tilde{\mathbf{e}} \in E$, then there exists $\tilde{E} \subset E : \tilde{\mathbf{e}} \in \tilde{E}$, but $(\mathbf{x} + \tilde{\mathbf{e}}) \notin A$, so the thesis follows.

(viii) and (ix) are particular cases of (vii). $\square$

The equality in the previous Proposition in (vi) is not true in general, as we can see from the following example:

Example 3.3 Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, $\lambda \in]0, 1[$.

$P_1 = (\frac{\sqrt{2}}{2} - \lambda, \frac{\sqrt{2}}{2} + \lambda^2)$,
$P_2 = (\frac{\sqrt{2}}{2} + \lambda^2, \frac{\sqrt{2}}{2} - \lambda)$. So $P_1, P_2 \in A$,
$E_1 = \{(x, y) \in \mathbb{R}^2 : x \geq \frac{\sqrt{2}}{2} - \lambda, y \geq \frac{\sqrt{2}}{2} + \lambda^2\}$,
$E_2 = \{(x, y) \in \mathbb{R}^2 : x \geq \frac{\sqrt{2}}{2} + \lambda^2, y \geq \frac{\sqrt{2}}{2} - \lambda\}$.

Then E_1 and E_2 are improvement sets and $E_1 \cap E_2 \subsetneq A$.

It turns out that $\mathbf{0} \notin O^{E_1}(A)$, $\mathbf{0} \notin O^{E_2}(A)$ but $\mathbf{0} \in O^{E_1 \cap E_2}(A)$. See Fig. 2.

Proposition 3.3 *If $A, B \subset \mathbb{R}^n$, then for each improvement set E the following relations hold*:

(i) $O^E(A) \cap O^E(B) \subset O^E(A \cap B)$.
(ii) $O^E(A \cap B) \subset O^E(A \cup B)$.
(iii) $O^E(A \cup B) \subset O^E(A) \cup O^E(B)$.

Proof (i) Let $\mathbf{x} \in O^E(A) \cap O^E(B)$ then $\mathbf{x} \in A$ and $\mathbf{x} \in B$, $\mathbf{x} \in A \cap B$ and $\mathbf{x} + \mathbf{e} \notin A$ and $\mathbf{x} + \mathbf{e} \notin B$ $\forall \mathbf{e} \in E$ then $\mathbf{x} + \mathbf{e} \notin A \cap B$ so $\mathbf{x} \in O^E(A \cap B)$.

(ii) Let $\mathbf{x} \in O^E(A \cap B)$ then $\mathbf{x} \in A \cap B$ and $\mathbf{x} + \mathbf{e} \notin A \cap B$ $\forall \mathbf{e} \in E$, then $\mathbf{x} \in A$ and $\mathbf{x} \in B$ so $\mathbf{x} \in A \cup B$, $\mathbf{x} + \mathbf{e} \notin A \cap B \subset A \cup B$ then $\mathbf{x} \in O^E(A \cup B)$.

(iii) Let $\mathbf{x} \in O^E(A \cup B)$, then $\mathbf{x} \in A \cup B$ and $\mathbf{x} + \mathbf{e} \notin A \cup B$ $\forall \mathbf{e} \in E$. If $\mathbf{x} \in A$, $\mathbf{x} + \mathbf{e} \notin A$ then $\mathbf{x} \in O^E(A)$, if $\mathbf{x} \in B$, $\mathbf{x} + \mathbf{e} \notin B$ then $\mathbf{x} \in O^E(B)$, so the thesis holds. Concluding:

$$O^E(A) \cap O^E(B) \subset O^E(A \cap B) \subset O^E(A \cup B) \subset O^E(A) \cup O^E(B). \qquad \square$$

Remark 3.3 Let $E \in \Im^2$: intuitively its boundary can be seen as a monotonic curve in $\mathbb{R}^2$ from "north-west to south-east" given by the parametrization: $K : \mathbb{R} \to \mathbb{R}^2$ where for each $t \in \mathbb{R}$ $K(t) = (K_1(t), K_2(t)) = (\beta(t) + t, \beta(t) - t)$ with $\beta(t) = \inf\{s \in \mathbb{R} : (s + t, s - t) \in E\}$.

Because of the upper comprehensiveness of E we have for $a < b$, $a, b \in \mathbb{R}$, $K_1(a) \leq K_1(b)$ and $K_2(a) \geq K_2(b)$.

Remark 3.4 We can also see the improvement set E as the graph of a (monotonic decreasing) multifunction

$$\mu : A \twoheadrightarrow 2^{\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}},$$

where $A =]a, +\infty[$ or $A = [a, +\infty[$, $a \in \mathbb{R} \cup \{-\infty\}$.

We can define μ in the following way: $\mu(x) = [\underline{\mu}(x), \overline{\mu}(x)]$, where $\underline{\mu}(x) = \inf\{y : (x, y) \in E\}$, $\overline{\mu}(x) = \sup\{y : (x, y) \in \overline{E}\} = \{+\infty\}$.

Then the graph of μ describes the improvement set E.

The multifunction μ is decreasing in the sense that $x_1 \leq x_2$ implies $\underline{\mu}(x_1) \geq \underline{\mu}(x_2)$ and $\mu(x_1) \subset \mu(x_2)$.

For multifunction properties see [22].

The following examples illustrate the previous concepts:

Example 3.4 Let $E = \mathbb{R}^2_{++}$, then $\mu : [0, +\infty[\twoheadrightarrow 2^{\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}}$ is such that: $\mu(0) = [0, +\infty)$ and $\mu(x) = [0, +\infty]$.

Example 3.5 Let be $E = \mathbb{R}^2_{++}$ or $E = \mathbb{R}^2_{+} \setminus \{\mathbf{0}\}$. In both cases the boundary of E is given by

$$E = \left\{(0, a) \in \mathbb{R}^2 : a > 0\right\} \cup \left\{(b, 0) \in \mathbb{R}^2 : b > 0\right\}.$$

The multifunction K of Remark 3.3 is

$$K(t)=\begin{cases}(0,-2t) & \text{for } t\in\,]-\infty,0],\\ (2t,0) & \text{for } t\in\,]0,+\infty[.\end{cases}$$

Remark 3.5 Let $E\subset\mathbb{R}^n$, $\mathbf{0}\notin E$ and for any $\mathbf{x}\in E$ any $h>0$ and for any $i=1,\dots,n$ it turns out $\mathbf{x}+h\mathbf{e}_i\in E$. Then E is an improvement set. (The converse is also true.)

In the following theorems we write some properties about improvement sets in relation to the epigraph of a decreasing function.

Theorem 3.1 *Let $E\subset\mathbb{R}^n$ $(n\geq 2)$ be a closed improvement set. Then there exists another improvement set $E_{-n}\subset\mathbb{R}^{n-1}$*

$$E_{-n}=\left\{(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1}:\exists(x_1,\dots,x_{n-1},x_n)\in E\right\}$$

and a function $\phi:E_{-n}\to\mathbb{R}\cup\{-\infty\}$, decreasing w.r.t. all its variables, such that E is the epigraph of ϕ w.r.t. the nth coordinate, that is:

$$E=\left\{(x_1,\dots,x_n)\in\mathbb{R}^n:\ x_n\geq\phi(x_1,\dots,x_{n-1})\ \forall\mathbf{x}_{-n}\in E_{-n}\right\}.$$

Proof From the hypotheses about E, $\mathbf{0}\notin E$ and we can choose an index j $(1\leq j\leq n)$ such that

$$\inf\left\{x_j:\ (x_1,\dots,x_n)\in E\right\}>0.$$

We can choose another index i $(1\leq i\leq n)$ such that $i\neq j$ $(n\geq 2$ by hypothesis). In order to make the notations easy, it is not a restriction to suppose, from now on, $i=n$. We claim that E_{-n} is an improvement set. First $\mathbf{0}\notin E_{-n}$ because, by definition of n, there exists an index (just j) such that $\inf\{x_j:(x_1,\dots,x_n)\in E\}>0$.

Second, by definition of E_{-n}, if $\mathbf{x}_{-n}=(x_1,\dots,x_{n-1})\in E_{-n}$, there exists a point $(x_1,\dots,x_i,\dots,x_n)\in E$. Since E is an improvement set, if $\mathbf{y}_{-n}=(y_1,\dots,y_{n-1})\geqq 0$, and $y_n\geq 0$, we also have $\mathbf{y}=(y_1,\dots,y_n)\geqq\mathbf{0}$. So $(x_1+y_1,\dots,x_n+y_n)\in E$. This proves that $\mathbf{x}_{-n}+\mathbf{y}_{-n}\in E_{-n}$ and E_{-n} is an improvement set.

Now we have to prove the existence of the function ϕ mentioned in the statement. For any $\mathbf{x}_{-n}\in\mathbb{R}^{n-1}$ such that there exists $(x_1,\dots,x_n)\in E$ let us define

$$\phi(\mathbf{x}_{-n}):=\inf\left\{x_n\in\mathbb{R}:\ (x_1,\dots,x_n)\in E\right\}.$$

We prove that E is the epigraphic of ϕ. For this goal we show that if $\mathbf{x}=(x_1,\dots,\overline{x}_n)\in E$ then $\mathbf{x}\in epi\,\phi$, i.e. $\overline{x}_n\geq\phi(\mathbf{x}_{-n})$. This is evident from the definition of ϕ.

Conversely, fix $\mathbf{x}_{-n}\in E_{-n}$; if $\phi(\mathbf{x}_{-n})\in\mathbb{R}$, then $(\mathbf{x}_{-n},\phi(\mathbf{x}_{-n}))\in E$. Besides, by the definition of infimum, we have $\overline{x}_n\geq\phi(\mathbf{x}_{-n})$ therefore, since E is an improvement set, it follows $(x_1,\dots,\overline{x}_n)\in E$ as we wanted.

If $\phi(\mathbf{x}_{-n})=-\infty$, there exists a number $\hat{x}_n$ such that $\hat{x}_n\leq\overline{x}_n$ and $(x_1,\dots,\hat{x}_n)\in E$; since $E\in\Im^n$, it turns out $(x_1,\dots,\overline{x}_n)\in E$. $\square$

Theorem 3.2 *Let $E_{-n} \subset \mathbb{R}^{n-1}$, $E_{-n} \in \Im^{n-1}$ and let $\phi : E_{-n} \to \mathbb{R} \cup \{-\infty\}$ be a decreasing function w.r.t. all its variables. Then the epigraph of ϕ:*

$$epi\,\phi := \left\{(x_1, \dots, x_n) \in \mathbb{R}^n : \mathbf{x}_{-n} \in E_{-n}, x_n \geq \phi(\mathbf{x}_{-n})\ \forall (x_1, \dots, x_n) \in \mathbb{R}^{n-1}\right\}$$

is an improvement set (for short $epi\,\phi \in \Im^n$).

Proof By hypothesis $\mathbf{0} \notin E_{-n}$, therefore ϕ cannot be defined in $\mathbf{0}$ and $\mathbf{0} \notin E$. Suppose now $\mathbf{x} = (x_1, \dots, x_n) \in epi\,\phi$ i.e. $x_n \geq \phi(\mathbf{x}_{-n})$ for some $\mathbf{x}_{-n} \in E_{-n}$. Given $h > 0$ it turns out that $x_n + h \geq \phi(\mathbf{x}_{-n})$ therefore $(\mathbf{x}_{-n}, x_n + h) \in epi\,\phi$. Consider now the case $1 \leq i \leq n - 1$ and $h > 0$. Since by hypothesis ϕ is decreasing w.r.t. all its variables, we have $\phi(\mathbf{x}_{-n} + h\mathbf{e}_i) \leq \phi(\mathbf{x}_{-n}) \leq x_n$ for all $i = 1, \dots, n - 1$, that is: $(\mathbf{x}_{-n} + h\mathbf{e}_i, x_n) \in epi\,\phi$ for all $i = 1, \dots, n - 1$ (in this context $\mathbf{e}_i \in \mathbb{R}^{n-1}$). So we conclude. □

Remark 3.6 From the previous theorems it follows that if the improvement set E is closed, the condition about $epi\,\phi$ becomes necessary and sufficient.

4 Existence of E-Optimal Points

Definition 4.1 The set $E \subset \mathbb{R}^n$ is *0-separable* iff there is a $\mathbf{w} \in \mathbb{R}^n_{++}$ such that $\langle \mathbf{w}.\mathbf{e} \rangle \geq 0$ for all $\mathbf{e} \in E$.

Definition 4.2 Let $\mathbf{p} \in \Delta$ and let $E \in \Im^n$. Then $\mathbf{p}$ is called a *separator* for E iff there exists a positive number t such that $\langle \mathbf{p}, \mathbf{e} \rangle > t$ for each $\mathbf{p} \in E$.

Intuitively, if $\mathbf{p}$ is a separator for E, there is a hyperplane H in $\mathbb{R}^n$ which strictly separates $\{\mathbf{0}\}$ from E:

$$H = \left\{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{p} \rangle = t\right\}.$$

Definition 4.3 Let $\mathbf{p} \in \Delta$ and $A \subset \mathbb{R}^n$. Then A is called *p-upper bounded* if $\exists\ k \in \mathbb{R}$ such that $\langle \mathbf{p}, \mathbf{a} \rangle \leq k$ for all $\mathbf{a} \in A$.

Intuitively A is contained in a hyper-half space with $\mathbf{p}$ as normal of the "boundary" hyperplane.

If $A \subset \mathbb{R}^n$ is upper bounded then A is $\mathbf{p}$-upper bounded for all $\mathbf{p} \in \Delta$.

In the following theorem we will give a sufficient condition for the existence of E-optimal points. To learn more about separation theorems see [23].

Theorem 4.1 *Let $E \in \Im^n$ and $A \subset \mathbb{R}^n$. Suppose that $\mathbf{p} \in \Delta$ is such that*

(i) $\mathbf{p}$ *is a separator of E and* $\mathbf{0}$;
(ii) *A is* $\mathbf{p}$*-upper bounded.*

Then $O^E(A) \neq \emptyset$.

Proof Let $\mathbf{p} \in \Delta$, $t > 0$, $k \in \mathbb{R}$ s.t. (i) and (ii) are valid.

Take $\hat{\mathbf{x}} \in A$ s.t. $\langle \mathbf{p}, \hat{\mathbf{x}} \rangle \geq \sup\{\langle \mathbf{p}.\mathbf{a} \rangle, \mathbf{a} \in A\} - t/2$. Then $\forall \mathbf{e} \in E$ it turns out that $\langle \mathbf{p}, \hat{\mathbf{x}} + \mathbf{e} \rangle \geq \sup\{\langle \mathbf{p}.\mathbf{a} \rangle : \mathbf{a} \in A\} - t/2 + t$ yielding $\langle \mathbf{p}, \hat{\mathbf{x}} + \mathbf{e} \rangle \geq \sup\{\langle \mathbf{p}.\mathbf{a} \rangle : \mathbf{a} \in A\} + t/2$.

Then

$$\langle \mathbf{p}, \hat{\mathbf{x}} + \mathbf{e} \rangle \notin pA \quad \Rightarrow \quad (\hat{\mathbf{x}} + \mathbf{e}) \notin A \quad \Rightarrow \quad (\hat{\mathbf{x}} + \mathbf{e}) \cap A = \emptyset. \qquad \square$$

Intuitively $\mathbf{p}$ has a double role: $\mathbf{p}A$ is contained in a half-hyperplane with normal $\mathbf{p}$ and there is a hyperplane with the same normal $\mathbf{p}$ which separates $\{\mathbf{0}\}$ from E.

Example 4.1 Let $A = \{(x, y) \in \mathbb{R}^2 : x \leq 0\}$ and the improvement set $E = \{(x, y) \in \mathbb{R}^2 : x + y \geq 1\}$.

Then $O^E(A) = \emptyset$, in fact the double role of the normal vector $\mathbf{p}$ is a necessary condition to have optimal points.

The strong separation of E from $\{\mathbf{0}\}$ is only a sufficient condition as we illustrate in the following

Example 4.2 Consider $E = \{(x, y) \in \mathbb{R}^2 : x + y \geq 0\} \setminus \{\mathbf{0}\}$ and $A = \{(x, y) \in \mathbb{R}^2 : x + y \leq -3\}$ then $O^E(A) = \emptyset$.

If $E = \mathbb{R}_{++}$, $A = \{(x, y) \in \mathbb{R}^2 : \ x \leq 0, \ y \leq 0, \ xy \geq 1\}$, E is not strongly separable from $\{\mathbf{0}\}$, but it turns out $O^E(A) = wPE(A) \neq \emptyset$.

Example 4.3 Consider $E = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \text{ or } x_2 > 0\}$ (the largest improvement set). E is not separable from $\mathbf{0}$.

If we take $A = \{\mathbf{a}, \mathbf{b}\}$ with $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ and not comparable, then $O^E(\{\mathbf{a}, \mathbf{b}\}) = \emptyset$.

5 Conclusion and Open Problems

In this paper, we have considered a new concept of optimal points for multicriteria situations keeping into account special sets E with two properties:

(i) $\mathbf{0} \notin E$ (exclusion property: that is $\mathbf{0}$ is not an essential improvement of itself) and
(ii) E is upper comprehensive i.e. $\mathbf{a} \in E$, $\mathbf{b} \geqq \mathbf{a}$ implies $\mathbf{b} \in E$ (comprehensive property, intuitively if $\mathbf{a}$ is an essential improvement of $\mathbf{0}$ and $\mathbf{b} \geqq \mathbf{a}$ then also $\mathbf{b}$ is an essential improvement of $\mathbf{0}$).

We defined an E-optimal point of the set A w.r.t. the improvement set E as that point $\mathbf{a} \in A$ s.t. $(\mathbf{a} + E) \cap A = \emptyset$.

This unifies the definitions of approximate and exact optimal solution and generalizes some which exists in literature.

We proved some properties about improvement sets and their relations with the epigraph of a decreasing function.

We studied topological properties about these new sets $O^E(A)$, we have given an existence theorem and many examples to illustrate this new concept.

Some questions about open problems arise:

(1) How can we extend this new concept of optimal points to equilibria for games? For a multi-leveled approach to games see [24].
(2) In [20], *well-posedness* properties for multicriteria situations were studied. How can we give new properties of well-posedness for E- optimal points?
(3) To restrict the set of E-optimal points, some refinements could be made.
(4) Is an axiomatic approach possible for the E-optimal points similar as that given for Pareto points in [25]?
(5) In [6], approximate Pareto points using cones K were defined. We can introduce K-comprehensive sets as those sets B where $\mathbf{b} + K \subset B$ for all $\mathbf{b} \in B$. Which results can we obtain generalizing Loridan's definition to K-comprehensive sets?
(6) If $A_n \to A$ with some opportune convergence, under which hypotheses $O^E(A_n) \to O^E(A)$. (About convergence of sets see [26, 27].)

Some of these issues are works in progress.

References

1. French, S.: Decision Theory: An Introduction to the Mathematics of Rationality. Ellis Horwood, New York (1993)
2. Stadler, W.: Initiations of multicriteria optimization. In: Jahn, J., Krabs, W. (eds.) Recent Advances and Historical Development of Vector Optimization, pp. 3–47. Springer, Berlin (1987)
3. Demyanov, V.F., Giannessi, F., Karelin, V.V.: Optimal control problems via exact penalty functions. J. Glob. Optim. **12**, 215–324 (1998)
4. Deng, S.: Boundedness and nonemptiness of the efficient solution sets in multiobjective optimization. J. Optim. Theory Appl. **144**(1), 29–42 (2010)
5. Sun, G.Z.: An economic approach to some classical theorems in optimization theory. Optim. Lett. **2**(2), 281–286 (2008).
6. Loridan, P.: Well posedness in vector optimization. In: Lucchetti, R., Revalski, J. (eds.) Recent Developments in Well Posed-Variational Problems, pp. 171–192. Kluwer Academic, Dordrecht (1995)
7. Gutierrez, C., Jiménez, B., Novo, V.: A unified approach and optimality conditions for approximate solutions of vector optimization problems. SIAM J. Optim. **17**(3), 688–710 (2006)
8. Gutierrez, C., Jiménez, B., Novo, V.: Optimality conditions via scalarization for a new ϵ-efficiency concept in vector optimization problems. Eur. J. Oper. Res. **201**, 11–22 (2010)
9. Gutierrez, C., Jiménez, B., Novo, V.: A generic approach to approximate efficiency and application to vector optimization with set-valued maps. J. Glob. Optim. **49**, 313–342 (2011)
10. Sawaragi, Y., Nakayama, H., Tanino, T.: Theory of Multiobjective Optimization. Academic Press, Orlando (1985)
11. Luc, D.T.: Theory of Vector Optimization. Lectures Notes in Economics and Mathematical Systems. Springer, Berlin (1989)
12. Di Pillo, G., Giannessi, F. (eds.): Nonlinear Optimization and Applications (1996)
13. Giannessi, F. (ed.): Vector Variational Inequalities and Vector Equilibria. Kluwer Academic, Dordrecht (2000)
14. Ehrgott, M.: Multicriteria Optimization, 2nd edn. Springer, Berlin (2005)
15. Chinchuluun, A., Pardalos, P.M.: A survey of recent developments in multiobjective optimization. Ann. Oper. Res. **154**, 29–50 (2007)
16. Chinchuluun, A., Pardalos, P.M., Migdalas, A., Pitsoulis, L.: Pareto Optimality, Game Theory and Equilibria. Edward Elgar, Cheltenham Glos (2008)
17. Pardalos, P.M., Rassias, T.M., Khan, A.A.: Nonlinear Analysis and Variational Problems. Springer, Berlin (2010)
18. Bednarczuck, E., Penot, J.P.: Metrically well-set minimization problems. Appl. Math. Optim. **26**, 273–285 (1992)
19. Bednarczuck, E.: An approach to well posedness in vector optimization: consequences to stability. Control Cybern. **23**, 107–122 (1994)

20. Morgan, J.: Approximations and well-posedness in multicriteria games. Ann. Oper. Res. **137**, 257–268 (2005)
21. Patrone, F., Pusillo, L., Tijs, S.: Multicriteria games and potentials. Top **15**, 138–145 (2007)
22. Klein, E., Thompson, A.: Theory of Correspondences. Wiley-Interscience, New York (1984)
23. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (1970).
24. Peters, H.: Game Theory—A Multileveled Approach. Springer, Berlin (2008)
25. Miglierina, E., Molho, E., Patrone, F., Tijs, S.: Axiomatic approach to approximate solutions in multiobjective optimization. Decis. Econ. Finance **31**(2), 95–115 (2008)
26. Kuratowski, C.: Topologie. Monografie Matematyczne, vol. 20. Polska Akad. Nauk, Warszawa (1952)
27. Beer, C.: On convergence of closed sets in a metric space and distance functions. Bull. Aust. Math. Soc. **31**, 421–431 (1985)