Harnack inequality for solutions of mixed boundary value problems containing boundary terms

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Abstract

We prove Harnack inequality near the boundary for solutions of mixed boundary value problems for a class of divergence form elliptic equations with discontinuous and unbounded coefficients, in presence of boundary integrals.

1 Introduction

In this note we prove Harnack inequality, near the boundary of Ω, of the solutions of a mixed problem for a class of divergence form elliptic equations, containing integral terms on the boundary.

Let Ω be an open subset Ω of $\mathbb{R}^n$, and V be the subspace of $H^1(\Omega)$ defined by

$$V := \{v \in H^1(\Omega) :\ v = 0 \text{ on } \Gamma_o \text{ in the sense of } H^1(\Omega)\} \tag{1}$$

where Γ_o is a closed (possibly empty) subset of $\partial\Omega$. Consider furthermore the bilinear form

$$a(u, v) := \int_\Omega \{\sum_{i,j=1}^{n} a_{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n}(b_i u_{x_i} v + d_i u v_{x_i}) + cuv\}\, dx + \int_\Gamma guv\, d\sigma \tag{2}$$

where $\Gamma := \partial\Omega \backslash \Gamma_o$.

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Let $u \in V$ be a solution of the equation

$$a(u,v) = \int_\Omega \{f_o v + \sum_{i=1}^n f_i v_{x_i}\}\,dx + \int_\Gamma hv\,d\sigma \quad \forall\ v \in V. \tag{3}$$

We can remark that, if the functions we consider are sufficiently regular (for example of class $C^1(\overline{\Omega})$), as well as Γ, then u is a solution of the following problem:

$$\begin{cases} Lu = f_o - \sum_{i=1}^n (f_i)_{x_i} \quad \text{in } \Omega, \\ \sum_{i,j=1}^n a_{ij} u_{x_i} N_j + \sum_{i=1}^n d_i N_i u + gu = h + \sum_{i=1}^n f_i N_i \quad \text{on } \Gamma, \\ u = 0 \quad \text{on } \Gamma_o \end{cases}$$

where N is the normal unit vector to Γ (oriented towards the exterior of Ω) and L the operator defined by

$$Lu := -\sum_{i,j=1}^n a_{ij} u_{x_i x_j} + \sum_{i=1}^n [b_i - d_i - \sum_{j=1}^n (a_{ij})_{x_j}] u_{x_i} + [c - \sum_{i=1}^n (d_i)_{x_i}] u \tag{4}$$

In a former work [1] we proved Hölder regularity up to the boundary for solutions of equation (3), under suitable hypotheses on the coefficients of the bilinear form $a(.,.)$ and known terms f_i $(i = 0, 1, 2, \ldots, n)$ and h.

In this note we prove Harnack inequality near the part Γ of the boundary for solutions of equation (3), following, as in [1], the classical work by Stampacchia [4].

2 Notations and hypotheses

Let Ω be an open bounded subset of $\mathbb{R}^{\mathrm{n}}$ (with $n \geq 3$ for simplicity); for the definition of the spaces $H^{1,p}(\Omega)$ we refer for example to [2], [3].

Let us suppose $a_{ij} \in L^\infty(\Omega)$ $(i,j = 1, 2, \ldots, n)$, $\sum a_{ij} t_i t_j \geq \nu |t|^2$ $\forall t \in \mathbb{R}^{\mathrm{n}}$ a.e. in Ω, with ν a positive constant; $b_i \in L^n(\Omega)$, $d_i \in L^p(\Omega)$, $(i = 1, 2, \ldots, n)$, $c \in L^{p/2}(\Omega)$, $g \in L^{\overline{p}}(\Gamma)$ with $p > n$, $\overline{p} := p(n-1)/n$.

Let Γ_o be a closed (possibly empty) subset of $\partial\Omega$, and define $\Gamma := (\partial\Omega)\backslash\Gamma_o$. We suppose that Γ is representable locally as the graph of a Lipschitz function: for the precise hypotheses see [1], par. 3.

3 Main results

Theorem 3.1 *Suppose that all the hypotheses listed in the previous paragraph are verified; let $u \in V$ be a non negative solution of the equation*

$$a(u, v) = 0 \qquad \forall v \in V$$

Then there exist two positive constants K_1 and $\overline{r}$, depending on the coefficients of $a(.,.)$ and on the regularity of Γ, such that for any $\overline{x} \in \Gamma$ and any r with $0 < r \leq \overline{r}$ it turns out

$$\operatorname*{ess\,sup}_{\Omega(\overline{x},r)} u \leq K_1 \operatorname*{ess\,inf}_{\Omega(\overline{x},r)} u$$

where $\Omega(\overline{x}, r) := \{x \in \Omega : \ |x_i - \overline{x}_i| < r, \ i = 1, 2, \cdots, n\}$.

Proof. As in [1], it is convenient to follow the proof e.g. of Theorem 8.2 of [4], remarking only the differences due to the presence of the new term $\int_\Gamma guv \, d\sigma$ in the definition of the bilinear form $a(.,.)$ (see (2)).
Let us begin with the proof of Lemma 8.1 of [4]. Proceding in the same manner, we find in the right member of the last inequality (at the end of page 239) the new term $|p| \int_\Gamma |g|\alpha^2 v^2 \, d\sigma$. By using the inequality (33) of [1] we get at once

$$\int_{\Gamma\cap Q} |g|\alpha^2 v^2 \, d\sigma \leq K_4^2 ||g||_{L^{n-1}(\Gamma\cap Q)} ||(\alpha v)_x||^2_{L^2(\Omega\cap Q)} \tag{5}$$

where we have denoted for brevity $Q := \{x \in \mathbb{R}^n : \ |x_i - \overline{x}_i| < r, \ i = 1, 2, \cdots, n\}$ and $\alpha \in C_o^1(Q)$. Note that in (5) the quantity $||g||_{L^{n-1}(\Gamma\cap Q)}$ can be made as small as we like by a suitable choice of $\overline{r}$. Therefore, combining (5) and the proof of Lemma 8.1 in [4], we can bring the term $||\alpha v_x||^2_{L^2(\Omega\cap Q)}$ to the left member of the last inequality of page 239 of [4], so the conclusion follows as in [4].

As what concerns Lemma 8.2 of [4], even in this case we find in the inequality (8.8) the new term $\int_\Gamma |g|\alpha^2 \, d\sigma$. In order to evaluate this integral, we can use

(5) where we put $v = 1$. Since it is $\alpha(x) = 0$ if $x \notin Q(\overline{x}, 2\rho)$, $|\alpha_x| \leq 2/\rho$ (see [4]), we easily deduce

$$\int_\Gamma |g|\alpha^2 \, d\sigma \leq K_o ||g||_{L^{n-1}(\Gamma\cap Q)} \rho^{n-2}$$

where K_o depends only on K_4 and n; the rest of the proof continues as in [4].

Also the remaining part of the proof of Theorem 8.1 of [4] can be followed in our new situation. There are few differences, for example in the proof of Lemma 8.4 (page 241 at the end) we must use Theorem 2 of [1] instead of the Remark 5.1 of [4]. The simple details can be left to the reader. $\square$

Corollary 1 *Suppose that all the hypotheses listed in the previous paragraph are verified; let $u \in V$ be a non negative solution of the equation (3). Then there exist two positive constants K_2 and $\overline{r}$, depending on the coefficients of $a(.,.)$ and on the regularity of Γ, such that for any $\overline{x} \in \Gamma$ and any r with $0 < r \leq \overline{r}$ it turns out*

$$\begin{aligned} &\operatorname*{ess\,sup}_{\Omega(\overline{x},r)} u \leq K_2 \operatorname*{ess\,inf}_{\Omega(\overline{x},r)} u + \\ &+ K_2 \left(\sum_{i=1}^{n} ||f_1||_{L^p(\Omega)} + ||f_o||_{L^{np/(n+p)}(\Omega)} + ||h||_{L^{\overline{p}}(\Gamma)} \right) r^{1-n/p} \end{aligned}$$

Proof. We can proceed again as in [4] (Theorem 8.2), where we must use Lemma 3 of [1] instead of Theorem 5.2 of [4]. $\square$

Corollary 2 *Suppose that all the hypotheses listed in the previous paragraph are verified; let $u \in V$ be a non negative solution of the equation*

$$a(u, v) = 0 \qquad \forall v \in V$$

Let G be an open subset of Ω such that $\overline{G} \cap \partial\Omega \subset \Gamma$. Then there exists a positive constant K_3, depending on G, Γ and the coefficients of $a(.,.)$, such that it turns out

$$\operatorname*{ess\,sup}_G u \leq K_3 \operatorname*{ess\,inf}_G u$$

Proof. As in [4] it is possible to deduce our assertion from the previous Theorem, by covering $\overline{G}$ by a finite number of sufficiently small cubes. In particular we must cover $\overline{G} \cap \partial\Omega = \overline{G} \cap \Gamma$ by small cubes with centres on Γ, in such a way as to apply the Theorem to each of them, while we can apply Theorem 8.1 of [4] to the cubes contained in Ω. The details of the proof can be left to the reader. $\square$

References

[1] M. Chicco, M. Venturino: *Hölder regularity for solutions of mixed boundary value problems containing boundary terms*, Boll. Un. Mat. Ital. (to appear)

[2] E. Gagliardo: *Proprietà di alcune classi di funzioni in più variabili*, Ricerche Mat. **7** (1958), 102–137

[3] O. A. Ladyzhenskaya, N. N. Ural'tseva: *Linear and quasilinear elliptic equations*, Academic Press, New York (1968)

[4] G. Stampacchia: *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier, Grenoble **15** (1965), 189–258.