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A priori inequalities for solutions of mixed boundary-value problems in unbounded domains

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Abstract. We prove some a priori inequalities for solutions of mixed boundary-value problems for a class of divergence form elliptic equations with discontinuous and unbounded coefficients in unbounded domains.

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1. Introduction

Given an open set Ω in $\mathbb{R}^n$, bounded or unbounded, we consider the subspace V of $H^1(\Omega)$ defined by

$$V := \{v \in H^1(\Omega) : \ v = 0 \text{ on } \Gamma_o \text{ in the sense of } H^1(\Omega)\}, \tag{1}$$

where Γ_o is a closed (eventually empty) subset of $\partial\Omega$. Given the bilinear form

$$\begin{aligned} a(u, v) := \int_\Omega \Big\{ \sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n (b_i u_{x_i} v + d_i u v_{x_i}) + cuv \Big\} \, dx \\ + \int_\Gamma guv \, d\sigma \end{aligned} \tag{2}$$

(where $\Gamma := \partial\Omega \backslash \Gamma_o$), let $u \in V$ be a solution of the equation

$$a(u, v) = \int_\Omega \Big\{ f_o v + \sum_{i=1}^n f_i v_{x_i} \Big\} \, dx + \int_\Gamma hv \, d\sigma \quad \forall v \in V. \tag{3}$$

It is easy to remark that, if all the considered functions are supposed sufficiently regular (for example belonging to $C^1(\overline{\Omega})$) as well as the boundary of Ω (or at least its part Γ), it is possible to define the operator

$$Lu := -\sum_{i,j=1}^n a_{ij} u_{x_i x_j} + \sum_{i=1}^n \Big[b_i - d_i - \sum_{j=1}^n (a_{ij})_{x_j} \Big] u_{x_i} + \Big[c - \sum_{i=1}^n (d_i)_{x_i} \Big] u, \tag{4}$$

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hence the function u satisfies (3) if and only if it satisfies the following equalities:

$$\text{(3')} \qquad \begin{cases} Lu = f_o - \sum_{i=1}^{n} (f_i)_{x_i} \quad \text{in } \Omega, \\ \sum_{i,j=1}^{n} a_{ij} u_{x_i} N_j + \sum_{i=1}^{n} d_i N_i u + gu = h + \sum_{i=1}^{n} f_i N_i \quad \text{on } \Gamma, \\ u = 0 \quad \text{on } \Gamma_o, \end{cases}$$

where N denotes the normal unit vector to Γ, oriented towards outside Ω. The function u therefore is a solution of a "mixed boundary-value problem" for the uniformly elliptic operator L. Nevertheless the bilinear form (2) and the equation (3) make sense even if the coefficients and data of the bilinear form $a(.,.)$ are not regular.

The aim of the present note is to study the minimal hypotheses on the coefficients and data in order that the bilinear form $a(.,.)$ be bounded on $V \times V$ even in the case of unbounded Ω. By extending the results of [6] we prove also that, if λ is sufficiently large, the bilinear form $a(.,.) + \lambda(.,.)_{L^2(\Omega)}$ is coercive on $V \times V$, so that for the same values of λ the boundary-value problem

$$\text{(5)} \quad \begin{cases} a(u,v) + \lambda(u,v)_{L^2(\Omega)} = \int_\Omega \left\{ f_o v + \sum_{i=1}^{n} f_i v_{x_i} \right\} dx + \int_\Gamma hv \, d\sigma \quad \forall v \in V, \\ u \in V \end{cases}$$

has one and only one solution, u, for any choice of the functions f_i $(i = 0, 1, \ldots, n)$ and h in suitable $L^p(\Omega)$ spaces.

Finally we study the subsolutions of (3): by supposing $u \in H^1(\Omega)$ such that

$$\text{(6)} \quad a(u,v) \leq \int_\Omega \left\{ f_o v + \sum_{i=1}^{n} f_i v_{x_i} \right\} dx + \int_\Gamma hv \, d\sigma, \quad \forall v \in V, \ v \geq 0 \text{ in } \Omega,$$

we prove an a priori inequality of the type

$$\operatorname*{ess\,sup}_\Omega u \leq K_1 \max_{\Gamma_o} u + K_2 \|u\|_{H^1(\Omega)} + K_3,$$

where K_1 is a constant depending only on n and K_2, K_3 also depend on the coefficients of the bilinear form $a(.,.)$, on the data f_i $(i = 0, 1, \ldots, n)$, h and the regularity of the part Γ of $\partial\Omega$.

Therefore the present note may be considered as a sequel of [5], where the set Ω was supposed bounded and the functions g, h defined on Γ were zero. In the particular case $\Gamma = \emptyset$, that is $\Gamma_o = \partial\Omega$, again we find the results of [2].

2. Notations and hypotheses on the coefficients

Let Ω be an open subset of $\mathbb{R}^n$; for simplicity we suppose $n \geq 3$ (even if, with a few changes, it would be possible to extend the results to the case $n = 2$). We refer, for example, to [4], [6] for the definition of the spaces $H^{1,p}(\Omega)$; in $H^1(\Omega) := H^{1,2}(\Omega)$ we put, by definition,

$$\|u_x\|_{L^2(\Omega)} := \left\{ \sum_{i=1}^{n} \|u_{x_i}\|^2_{L^2(\Omega)} \right\}^{1/2},$$

where we assume as a norm, for instance, the quantity

$$\|u\|_{H^1(\Omega)} := \left\{ \|u\|^2_{L^2(\Omega)} + \|u_x\|^2_{L^2(\Omega)} \right\}^{1/2}.$$

Definition 1. *Let $p \geq 1,\ \delta > 0,\ f \in L^p_{loc}(\Omega)$; we define*

$$\omega(f, p, \delta) := \sup\{\|f\|_{L^p(E)} :\ E \text{ measurable}, E \subset \Omega,\ \text{meas } E \leq \delta\}$$

$$X^p(\Omega) := \left\{ f \in L^p_{loc}(\Omega) :\ \omega(f, p, \delta) < +\infty\ \ \forall \delta > 0 \right\}$$

$$X^p_o(\Omega) := \left\{ f \in X^p(\Omega) :\ \lim_{\delta \to 0+} \omega(f, p, \delta) = 0 \right\}.$$

For further properties of these spaces see [2].

Suppose now $a_{ij} \in L^\infty(\Omega)$ $(i, j = 1, 2, \ldots, n)$, $\sum a_{ij} t_i t_j \geq \nu |t|^2\ \forall t \in \mathbb{R}^n$ a.e. in Ω, with ν positive constant. Except for further hypotheses, we suppose at least $b_i, d_i \in X^n(\Omega)$ $(i = 1, 2, \ldots, n)$, $c \in X^{n/2}(\Omega)$, $g \in X^{n-1}(\Gamma)$.

If $u \in H^1(\Omega)$, $m \in \mathbb{R}$, B is a closed subset of $\overline{\Omega}$, we say that $u \leq m$ $(u = m)$ on B in the sense of $H^1(\Omega)$ if there exists a sequence of real functions $u_j \in C^1(\overline{\Omega}) \cap H^1(\Omega)(j = 1, 2, \ldots)$ such that $u_j \leq m$ $(u_j = m)$ in B for any $j \in \mathbb{N}$ and $\lim_j \|u - u_j\|_{H^1(\Omega)} = 0$.

3. Hypotheses on the boundary of Ω

Let us suppose that there exists an open set $\Omega_1 \supset \Omega$ with the following properties. Let $\partial\Omega_1$ be "locally uniformly Lipschitz" (such a definition will be made precise later). Furthermore we define

$$\Gamma := (\partial\Omega_1) \cap (\partial\Omega), \qquad \Gamma_o := \overline{(\partial\Omega) \backslash (\partial\Omega_1)}.$$

With these choices, if $u \in H^1(\Omega)$, $u = 0$ on Γ_o in the sense of $H^1(\Omega)$, and we prolong u to be zero in $\Omega_1 \backslash \Omega$, it turns out that $u \in H^1(\Omega_1)$ still (where we have also denoted by u the prolonged function). Please note that, under our hypotheses, while Γ is supposed sufficiently regular, the part Γ_o of the boundary of Ω may be irregular.

Now we formulate the hypothesis on the regularity of $\partial\Omega_1$ (and therefore on Γ). We suppose that there exist two positive numbers $\overline{r}$, K such that for every point

$\overline{x} \in \partial\Omega_1$ it is possible to find an n-dimensional cube (with a suitable choice of the cartesian axes) such as

$$Q(\overline{x}, \overline{r}) := \left\{x \in \mathbb{R}^n : \ |x - \overline{x}_i| < \overline{r}, \ i = 1, 2, \dots, n\right\} \tag{7}$$

having the following properties. Let us denote by D the $(n-1)$-dimensional cube

$$D := \left\{y \in \mathbb{R}^{n-1} : \ |y_i - \overline{x}_i| < \overline{r}, \ i = 1, 2, \dots, n-1\right\} \tag{8}$$

and suppose that there exists a function $\phi : \ D \to \mathbb{R}$ such that

$$\phi(\overline{x}_1, \overline{x}_2, \dots, \overline{x}_{n-1}) = \overline{x}_n, \tag{9}$$

$$\begin{aligned} &Q(\overline{x}, \overline{r}) \cap \Omega_1 \\ &\quad = \left\{y \in \mathbb{R}^n : \ (y_1, y_2, \dots, y_{n-1}) \in D, \ y_n < \phi(y_1, y_2, \dots, y_{n-1})\right\} \end{aligned} \tag{10}$$

$$\begin{aligned} &Q(\overline{x}, \overline{r}) \cap \partial\Omega_1 \\ &\quad = \left\{y \in \mathbb{R}^n : \ (y_1, y_2, \dots, y_{n-1}) \in D, \ y_n = \phi(y_1, y_2, \dots, y_{n-1})\right\} \end{aligned} \tag{11}$$

$$|\phi(y') - \phi(y'')| \le K|y' - y''| \ \ \forall y', y'' \in D. \tag{12}$$

Let us consider now, instead of D, its subset (with $0 < \delta \le 1$)

$$D_\delta := \left\{y \in \mathbb{R}^{n-1} : \ |y_i - \overline{x}_i| < \delta\overline{r}, \ i = 1, 2, \dots, n-1\right\}. \tag{13}$$

From (9), (12), (13) it follows that

$$\left|\phi(y') - \overline{x}_n\right| \le K\left|y' - \overline{y}\right| < K\sqrt{n-1}\,\delta\overline{r} \ \ \forall y' \in D_\delta, \tag{14}$$

where we have put, for brevity, $\overline{y} := (\overline{x}_1, \overline{x}_2, \dots, \overline{x}_{n-1}) \in \mathbb{R}^{n-1}$. Now let us choose

$$\delta := \min\left\{1/2, \ 1/(2K\sqrt{n-1})\right\} \tag{15}$$

so that from (14) it follows that

$$\left|\phi(y') - \overline{x}_n\right| < \overline{r}/2 \ \ \forall y' \in D_\delta. \tag{16}$$

From now on, besides the cubes $Q(\overline{x}, \overline{r})$ defined in (7), we shall also consider the n-dimensional parallepipeds defined as follows:

$$Q_\delta(\overline{x}, \overline{r}) := \left\{y \in \mathbb{R}^n : \ |y_i - \overline{x}_i| < \delta\overline{r} \ (i = 1, 2, \dots, n-1), \ |y_n - \overline{x}_n| < \overline{r}\right\}, \tag{17}$$

where δ is always defined by (15). Since we have supposed $0 < \delta \le 1/2$, it turns out that

$$Q(\overline{x}, \delta\overline{r}) \subset Q_\delta(\overline{x}, \overline{r}) \subset Q(\overline{x}, \overline{r}). \tag{18}$$

Please note that, while the function ϕ may change with the point $\overline{x} \in \partial\Omega_1$ and the cube $Q(\overline{x}, \overline{r})$, the constants $\overline{r}$ and K do not change (by hypothesis).

We can also remark that, if the preceding hypothesis is satisfied for some $\overline{r} > 0$, it is also satisfied if we replace $\overline{r}$ by any (positive) smaller number; therefore it is possible to choose $\overline{r}$ as small as we please (it will be fixed suitably later). Under these hypotheses, in almost every point of the boundary of Ω_1 there exists the normal unit vector to $\partial\Omega_1$, which we suppose is oriented towards the exterior of Ω_1. Such a normal unit vector will be denoted by N.

Lemma 1. *For any $x \in \overline{\Omega}$ there exist two cubes $Q(\overline{x}, r)$, $Q(\overline{x}, \theta r)$ such that $x \in Q(\overline{x}, r)$ and*

$$\|u\|_{L^{2^*}(Q(\overline{x},r)\cap\Omega)} \leq (K_5/r)\|u\|_{L^2(Q(\overline{x},\theta r)\cap\Omega)} + K_5\|u_x\|_{L^2(Q(\overline{x},\theta r)\cap\Omega)},$$

for any $u \in V$, where K_5 and θ are constants depending only on K and n, not depending on u nor on r, and $2^* := 2n/(n-2)$.

Proof. Let r be a positive number and consider a countable family of cubes

(19)
$$Q_h = Q(x_h, r) := \{y \in \mathbb{R}^n : |x_{hi} - y_i| < r \ (i = 1, 2, \ldots, n)\} \ (h = 1, 2, \ldots)$$

such that

$$\cup_{h=1}^{+\infty}\overline{Q_h} = \mathbb{R}^n, \qquad Q_h \cap Q_k = \emptyset \text{ if } h \neq k \ (h, k \in \mathbb{N}). \tag{20}$$

Furthermore let us suppose that

$$r = \delta\overline{r}/(2\sqrt{n}) = \min\{\overline{r}/(4\sqrt{n}), \overline{r}/[4K\sqrt{n(n-1)}]\}, \tag{21}$$

where $\overline{r}$ is the number connected with the regularity of $\partial\Omega_1$ (as introduced at the beginning of this paragraph) and δ defined by (15). Let us denote by N_1 the subset of $\mathbb{N}$ containing the indices $k \in \mathbb{N}$ such that $Q_k \subset \Omega$, i.e.

$$N_1 := \{k \in \mathbb{N} : Q_k \subset \Omega\},$$

and, in a similar way, define

$$N_2 := \{k \in \mathbb{N} : Q_k \cap \Gamma_o \neq \emptyset, \ Q_k \cap \Gamma = \emptyset\}$$
$$N_3 := \{k \in \mathbb{N} : Q_k \cap \Gamma \neq \emptyset\}.$$

Since $\overline{\Omega} = \Omega \cup \Gamma \cup \Gamma_o$ and taking (20) into account, from these definitions it follows that

$$\overline{\Omega} \subset (\cup_{k\in N_1}\overline{Q_k}) \cup (\cup_{k\in N_2}\overline{Q_k}) \cup (\cup_{k\in N_3}\overline{Q_k}). \tag{22}$$

The cubes Q_k ($k \in \mathbb{N}$) will be treated in a different way if either $k \in N_1$ or $k \in N_2$ or $k \in N_3$.

The simplest case occurs when $Q_k \subset \Omega$ (case 1); in this case, if $u \in V$, the restriction of u to Q_k clearly belongs to $H^1(Q_k)$, therefore we can directly apply Lemma 2 of [3] to such a cube. If x belongs to one of such cubes Q_k, the lemma is proved with $K_5 = 2^{(3n-4)/(n-2)}$, $\theta = 1$.

Even the case 2, in which $Q_k \cap \Gamma = \emptyset$, $Q_k \cap \Gamma_o \neq \emptyset$ is rather simple. According to our hypotheses, if $u \in V$ it follows that $u = 0$ on Γ_o in the sense of $H^1(\Omega)$, so we can prolong the definition of u to all of Q_k defining it as equal to zero in $Q_k \backslash \Omega$ (in fact by hypothesis we have $(\partial\Omega) \cap Q_k \subset \Gamma_o$). The function prolonged in this way, still denoted by u, belongs to $H^1(Q_k)$ and again we can apply to it Lemma 2 of [3]. So even in this case, if x belongs to the cube Q_k, the lemma is proved with the same constants as before.

Only the case 3 remains, in which $Q_k \cap \Gamma \neq \emptyset$. Choose a point $\overline{x}_k \in Q_k \cap \Gamma$ and consider the cube $Q(\overline{x}_k, \overline{r})$ which exists according to the hypotheses on Γ at the beginning of paragraph 3. Please note that the cube $Q(\overline{x}_k, \overline{r})$ may be oriented differently from the cube $Q(x_k, r)$, but from (18) and (21) it turns out that $Q_k \subset Q(\overline{x}_k, \delta\overline{r})$, whence

$$Q_k \subset Q_\delta(\overline{x}_k, \overline{r}) \subset Q(\overline{x}_k, \overline{r}). \tag{23}$$

Note also that, since $\overline{x}_k \in Q_k$, each cube of the family $\{Q_h\}_{h\in\mathbb{N}}$ has a non-empty intersection with at most a finite number of cubes of the family $\{Q(\overline{x}_k, \overline{r})\}_{k\in N_3}$, and conversely each cube of the family $\{Q(\overline{x}_k, \overline{r})\}_{k\in N_3}$ has a non-empty intersection with at most a finite number of cubes of the family $\{Q_h\}_{h\in\mathbb{N}}$.

More precisely, according to (21) we deduce what follows. If we fix $h \in \mathbb{N}$, it turns out that

$$Q_h(x_h, r) \cap Q(\overline{x}_k, \overline{r}) \neq \emptyset,$$

at most for a finite number $\overline{n}$ of indexes $k \in N_3$. Such a number $\overline{n}$ does neither depend on the cube $Q_h = Q_h(x_h, r)$ we started from, nor on r and $\overline{r}$, but only on the ratio $\overline{r}/r$ (which, according to (21), depends only on n and K). It follows that, given any function $f \in L_1(\Omega)$ (where we have defined $\widetilde{Q}_k := Q(\overline{x}_k, \overline{r})$),

$$\begin{aligned} \sum_{k\in N_3} \int_{\Omega\cap\widetilde{Q}_k} |f|\,dx &= \sum_{k\in N_3} \sum_{h\in\mathbb{N}} \int_{\Omega\cap\widetilde{Q}_k\cap Q_h} |f|\,dx \\ &= \sum_{h\in\mathbb{N}} \left(\sum_{k\in N_3} \int_{\Omega\cap\widetilde{Q}_k\cap Q_h} |f|\,dx \right) \le \overline{n} \sum_{h\in\mathbb{N}} \int_{\Omega\cap Q_h} |f|\,dx = \overline{n} \int_\Omega |f|\,dx. \end{aligned} \tag{24}$$

Therefore, if $u \in H^1(\Omega)$, we can apply Sobolev inequalities to the function u and to the sets $\Omega_1 \cap Q_\delta(\overline{x}_k, \overline{r})$ instead of $\Omega_1 \cap Q_k$, by using the hypothesis on Γ described before. Consider, in fact, the following change of variables:

$$\begin{cases} Y_i = \overline{x}_i + (y_i - \overline{x}_i)/\delta \ (i = 1, 2, \ldots, n-1) \\ Y_n = \overline{x}_n - \overline{r} + 2\overline{r}(y_n - \overline{x}_n + \overline{r})/[\phi(y_1, y_2, \ldots, y_{n-1}) - \overline{x}_n + \overline{r}], \end{cases} \tag{25}$$

where $\overline{x},\ \overline{r},\ \phi$ were defined in paragraph 3 (hypothesis on Γ). It is clear that if $y_n = \overline{x}_n - \overline{r}$ is also $Y_n = \overline{x}_n - \overline{r}$, while if $y_n = \phi(y_1, y_2, \ldots, y_{n-1})$ it turns out that $Y_n = \overline{x}_n + \overline{r}$, and if $y_i - \overline{x}_i = \pm\,\delta\overline{r}$ $(i = 1, 2, \ldots, n-1)$ it follows that $Y_i - \overline{x}_i = \pm\,\overline{r}$. We remark also that, by (16), if $(y_1, y_2, \ldots, y_{n-1}) \in D_\delta$ the denominator in the second member of (25) is greater than $\overline{r}/2$. Therefore, recalling (9), (10), (11), the change of variables (25) transforms the set

$$\begin{aligned} &Q_\delta(\overline{x}, \overline{r}) \cap \Omega_1 \\ &= \big\{ y \in \mathbb{R}^n :\ (y_1, y_2, \ldots, y_{n-1}) \in D_\delta,\ \overline{x}_n - \overline{r} < y_n < \phi(y_1, y_2, \ldots, y_{n-1}) \big\} \end{aligned}$$

in the following:

$$\widetilde{Q}(\overline{x}, \overline{r}) := \big\{ Y \in \mathbb{R}^n :\ |Y_i - \overline{x}_i| < \overline{r}\ \ (i = 1, 2, \ldots, n) \big\}.$$

We also note that the change of variables (25) is clearly invertible between these two sets.

Taking into account the hypothesis on ϕ (Lipschitz continuity) and the change of variables (25), it is easy to check that for almost all $y' \in D$ it turns out that

$$\left|\frac{\partial Y_n}{\partial y_n}\right| \le 4, \ \left|\frac{\partial Y_n}{\partial y_i}\right| \le 16K, \ \left|\frac{\partial Y_i}{\partial y_i}\right| \le 1/\delta, \ (i = 1, 2 \ldots, n-1), \tag{26}$$

where K is the Lipschitz constant of ϕ and δ is defined by (15). We can therefore apply Lemma 2 of [3] to the cube $\widetilde{Q} := Q(\overline{x}, \overline{r})$ and obtain

$$\|u\|_{L^{2^*}(\widetilde{Q})} \le (K_4/\overline{r})\|u\|_{L^2(\widetilde{Q})} + K_4\|u_Y\|_{L^2(\widetilde{Q})}, \tag{27}$$

where $2^* := 2n/(n-2)$, K_4 is a constant depending only on n and we have still denoted by u the function expressed in the new variables (25). By taking into account the theorem of integration by substitution, Theorem 1.V of [4] and (26), (27) we get

$$\|u\|_{L^{2^*}(\widehat{Q}\cap\Omega)} \le (K_5/\overline{r})\|u\|_{L^2(\widehat{Q}\cap\Omega)} + K_5\|u_x\|_{L^2(\widehat{Q}\cap\Omega)} \tag{28}$$

in which we have put, for brevity, $\widehat{Q} := Q_\delta(\overline{x}, \overline{r})$ and the constant K_5 depends only on K and n (note that δ, because of (15), depends on the same quantities). Remembering (18) and (21), if $x \in Q_k$, the lemma follows from (28) in this case also, with $\theta = \overline{r}/r\delta = 2\sqrt{n}/\delta$. □

The preceding inequalities lead us to the following proposition, which is well known (see for example Theorem 5.I of [4] and the remark in paragraph 7, or [1] Ch. 5). Nevertheless we think it may be useful to insert it, since with our method it is possible to evaluate exactly the constants which appear in it.

Proposition. *There exists a constant K_6 depending on n, K and on the set Γ, such that*

$$\|u\|_{L^{2^*}(\Omega)} \le K_6\|u\|_{H^1(\Omega)} \quad \forall u \in V, \tag{29}$$

where we have defined $2^* := 2n/(n-2)$.

Proof. Let $w := u\|u\|^{-1}_{H^1(\Omega)}$, evidently it suffices to prove that

$$\|w\|_{L^{2^*}(\Omega)} \le K_6. \tag{30}$$

Let $N_1, \ N_2, \ N_3$ be the subsets of $\mathbb{N}$ defined before. We have

$$\begin{aligned} &\int_\Omega |w|^{2^*}\,dx \\ &= \sum_{k\in N_1}\int_{\Omega\cap Q_k}|w|^{2^*}\,dx + \sum_{k\in N_2}\int_{\Omega\cap Q_k}|w|^{2^*}\,dx + \sum_{k\in N_3}\int_{\Omega\cap Q_k}|w|^{2^*}\,dx. \end{aligned} \tag{31}$$

If $k \in N_1$ or $k \in N_2$ from Lemma 2 of [3] we deduce that

$$\|w\|_{L^{2^*}(\Omega\cap Q_k)} \le (K_4/r)\|w\|_{L^2(\Omega\cap Q_k)} + K_4\|w_x\|_{L^2(\Omega\cap Q_k)}, \tag{32}$$

if instead $k \in N_3$ from (23) and (28) one gets

$$\|w\|_{L^{2^*}(\Omega\cap Q_k)} \le (K_5/\overline{r})\|w\|_{L^2(\Omega\cap\widetilde{Q}_k)} + K_5\|w_x\|_{L^2(\Omega\cap\widetilde{Q}_k)}, \tag{33}$$

where $\widetilde{Q}_k := Q(\overline{x}_k, \overline{r})$. Hence, also taking into account (24),

$$\begin{aligned}\sum_{k\in N_1\cup N_2\cup N_3} \|w\|^{2^*}_{L^{2^*}(\Omega\cap Q_k)} &\le \sum_{k\in N_1\cup N_2\cup N_3} \|w\|^{2}_{L^{2^*}(\Omega\cap Q_k)}\\ &\le 2\overline{n}K_5^2\left[\left(1/\overline{r}^2\right)\|w\|^2_{L^2(\Omega)} + \|w_x\|^2_{L^2(\Omega)}\right] \le 2\overline{n}K_5^2\left[1+\left(1/\overline{r}^2\right)\right].\end{aligned}$$

From this inequality and (31) we get at once (30), that is the assertion, with $K_6 = \{2\overline{n}K_5^2(1+1/\overline{r}^2)\}^{1/2^*}$. □

4. Integral inequalities on Γ

Since in the bilinear form $a(.,.)$ we now also have an integral on Γ, we must get some Sobolev-type inequalities suitable to treat it. It is sufficient to use again Lemma 5.II of Gagliardo [4], where we put $r = 1$, $m = n-1$, $p = s$, with $1 < s < n$. As in [3], we directly follow the proof by Gagliardo, replacing Ω with the cube $Q(\overline{x}, \overline{r})$ and supposing temporarily that $u \in H^1(Q(\overline{x}, \overline{r}))$. With simple calculations we get

$$\begin{aligned}&\left\{\int_{D_i} |u|^{s(n-1)/(n-s)}\, d\sigma\right\}^{(n-s)/s(n-1)}\\ &\le (K_7/\overline{r})\|u\|_{L^s(Q(\overline{x},\overline{r}))} + K_8\|u_x\|_{L^s(Q(\overline{x},\overline{r}))},\end{aligned} \tag{34}$$

where

$$\begin{cases} K_7 := 1 + 2^{(ns-2s+n)/(n-s)},\\ K_8 := 5s(n-1)/(n-s)\end{cases}$$

and D_i is any $(n-1)$-dimensional face of the cube $Q(\overline{x}, \overline{r})$. But formula (34) is not directly useful since we need to calculate the integral not on the face D_i of the cube, but on the "curved" intersection $\Gamma \cap Q(\overline{x}, \overline{r})$. Nevertheless, from the proof of the lemma of Gagliardo we see that we can rewrite (34) in the more precise form

$$\begin{aligned}&\left\{\int_{D_i} \sup_{y_i} |u|^{s(n-1)/(n-s)}\, d\sigma\right\}^{(n-s)/s(n-1)}\\ &\le (K_7/\overline{r})\|u\|_{L^s(Q(\overline{x},\overline{r}))} + K_8\|u_x\|_{L^s(Q(\overline{x},\overline{r}))},\end{aligned} \tag{35}$$

where we have put

$$\sup_{y_i} |u| := \sup\{|u(y_1, y_2, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_n)| : \overline{x}_i - \overline{r} \le t \le \overline{x}_i + \overline{r}\}.$$

By taking into account (35) and the definition of the surface integral, we deduce that

$$\int_{Q(\overline{x},\overline{r})\cap\Gamma} |u|^{s(n-1)/(n-s)}\, d\sigma$$

$$= \int_{D_i} |u(y_1, y_2, \ldots, y_{n-1}, \phi(y_1, y_2, \ldots, y_{n-1})|^{s(n-1)/(n-s)}$$

$$\times \sqrt{1 + \sum_{i=1}^{n-1} (\partial\phi/\partial y_i)^2}\, dy_1 dy_2 \ldots dy_{n-1} \tag{36}$$

$$\leq \sqrt{1 + (n-1)K^2} \int_{D_i} \sup_{y_i} |u|^{s(n-1)/(n-s)}\, d\sigma$$

$$\leq \sqrt{1 + (n-1)K^2} \left[(K_7/\overline{r})\|u\|_{L^s(Q(\overline{x},\overline{r}))} + K_8\|u_x\|_{L^s(Q(\overline{x},\overline{r}))}\right]^{s(n-1)/(n-s)},$$

whence the conclusion immediately follows. □

In the same context we can prove:

Lemma 2. *Let $g \in X^{n-1}(\Gamma)$, $u, v \in H^1(\Omega)$. Then there exists a constant K_9 depending on only g, n, K, Γ, such that*

$$\left| \int_\Gamma guv\, d\sigma \right| \leq K_9 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \tag{37}$$

Proof. The inequality may be proved by using (36) where we put $s = 2$. Recalling (23) also, we have (where $k \in N_3$, $\widehat{Q}_k := Q_\delta(\overline{x}_k, \overline{r})$, $\widetilde{Q}_k := Q(\overline{x}_k, \overline{r})$ and $\widehat{2} := (2n-2)/(n-2)$):

$$\begin{aligned} \left| \int_{\Gamma\cap\widehat{Q}_k} guv\, d\sigma \right| &\leq \|g\|_{L^{n-1}(\Gamma\cap\widetilde{Q}_k)} \|u\|_{L^{\widehat{2}}(\Gamma\cap\widetilde{Q}_k)} \|v\|_{L^{\widehat{2}}(\Gamma\cap\widetilde{Q}_k)} \\ &\leq K_{10}\|g\|_{L^{n-1}(\Gamma\cap\widetilde{Q}_k)} \left[(1/r)\|u\|_{L^2(\widetilde{Q}_k\cap\Omega)} + \|u_x\|_{L^2(\widetilde{Q}_k\cap\Omega)}\right] \\ &\quad \times \left[(1/r)\|v\|_{L^2(\widetilde{Q}_k\cap\Omega)} + \|v_x\|_{L^2(\widetilde{Q}_k\cap\Omega)}\right] \\ &\leq K_{10}\omega(g, n-1, \sqrt{1+(n-1)K^2}(2\overline{r})^{n-1}) \\ &\times \left[(1/r^2)\left(\|u\|^2_{L^2(\Omega\cap\widetilde{Q}_k)} + \|v\|^2_{L^2(\Omega\cap\widetilde{Q}_k)}\right) + \|u_x\|^2_{L^2(\Omega\cap\widetilde{Q}_k)} + \|v_x\|^2_{L^2(\Omega\cap\widetilde{Q}_k)}\right], \end{aligned} \tag{38}$$

where K_{10} is a constant depending only on n, K_7, K_8 and therefore finally only on n and K. Hence, by summing with respect to k in N_3 and taking into account (24) we get

$$\begin{aligned} \left| \int_\Gamma guv\, d\sigma \right| &\leq \overline{n} K_{10}\omega\left(g, n-1, \sqrt{1+(n-1)K^2}(2\overline{r})^{n-1}\right) \\ &\times \left[(1/r^2)\left(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\right) + \|u_x\|^2_{L^2(\Omega)} + \|v_x\|^2_{L^2(\Omega)}\right], \end{aligned} \tag{39}$$

whence the conclusion (37). □

5. Properties of the bilinear form $a(.,.)$

The a priori inequalities of Sobolev type we have illustrated in the former paragraph allow us to extend to our situation some properties of the bilinear form $a(.,.)$ already proved by Stampacchia [6] for the Dirichlet problem in an open bounded set.

Theorem 1. *Suppose that the hypotheses mentioned above are satisfied. Then the bilinear form $a(.,.)$ defined by (2) is bounded on $V \times V$, that is there exists a constant K_{11} depending on the coefficients of $a(.,.)$, on n and on Γ, such that*

$$|a(u,v)| \leq K_{11}\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \quad \forall\, u, v \in V. \tag{40}$$

Proof. It is sufficient to give suitable upper bounds to the various terms of the bilinear form $a(.,.)$ as defined in (2). We have

$$\Big|\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}u_{x_i}v_{x_j}\,dx\Big| \leq K_{12}\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \tag{41}$$

where K_{12} is a constant depending only on n and on $\max\{\|a_{ij}\|_{L^\infty(\Omega)} : i, j = 1, 2, \ldots, n\}$.

Now let $Q(x_k, r)$ be one of the cubes defined in (19), with $k \in N_1 \cup N_2$. From the Schwartz–Hölder inequality and Lemma 2 of [3] we have

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\int_{Q_k\cap\Omega} b_i u_{x_i} v\,dx\right| \leq \sum_{i=1}^{n}\|b_i\|_{L^n(Q_k\cap\Omega)}\|u_x\|_{L^2(Q_k\cap\Omega)}\|v\|_{L^{2^*}(Q_k\cap\Omega)} \\
&\leq \sum_{i=1}^{n}\|b_i\|_{L^n(Q_k\cap\Omega)}\|u_x\|_{L^2(Q_k\cap\Omega)}\left[(K_4/r)\|v\|_{L^2(Q_k\cap\Omega)} + K_4\|v_x\|_{L^2(Q_k\cap\Omega)}\right] \\
&\leq \sum_{i=1}^{n}\|b_i\|_{L^n(Q_k\cap\Omega)}\left[\|u_x\|^2_{L^2(Q_k\cap\Omega)} + (K_4/r)^2\|v\|^2_{L^2(Q_k\cap\Omega)} + K_4^2\|v_x\|^2_{L^2(Q_k\cap\Omega)}\right] \\
&\leq \sum_{i=1}^{n}\omega(b_i, n, (2r)^n)\left[\|u_x\|^2_{L^2(Q_k\cap\Omega)} + (K_4/r)^2\|v\|^2_{L^2(Q_k\cap\Omega)} + K_4^2\|v_x\|^2_{L^2(Q_k\cap\Omega)}\right],
\end{aligned} \tag{42}$$

where K_4 is the constant of Lemma 2 of [3], which depends only on n.

Furthermore let $Q_\delta(\overline{x}_k, \overline{r})$ be one of the parallelepipeds introduced by (17). It is possible to write an inequality similar to (42) by applying (28) instead of Lemma 2 of [3]. So we get (where we put, for brevity, $\widehat{Q}_k := Q_\delta(\overline{x}_k, \overline{r})$):

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\int_{\widehat{Q}_k\cap\Omega} b_i u_{x_i} v\,dx\right| \leq \cdots \\
&\leq \sum_{i=1}^{n}\omega\big(b_i, n, (2\overline{r})^n\big)\left[\|u_x\|^2_{L^2(\widehat{Q}_k\cap\Omega)} + (K_5/\overline{r})^2\|v\|^2_{L^2(\widehat{Q}_k\cap\Omega)} + K_5^2\|v_x\|^2_{L^2(\widehat{Q}_k\cap\Omega)}\right].
\end{aligned} \tag{43}$$

From (22), (23), (42), (43) we deduce at once that

$$
\begin{aligned}
(44)\quad & \Big|\sum_{i=1}^{n}\int_{\Omega} b_i u_{x_i} v\,dx\Big| \\
&\le \sum_{k\in N_1\cup N_2}\Big|\int_{\Omega\cap Q_k}\sum_{i=1}^{n} b_i u_{x_i} v\,dx\Big| + \sum_{k\in N_3}\Big|\int_{\Omega\cap \widehat{Q}_k}\sum_{i=1}^{n} b_i u_{x_i} v\,dx\Big| \\
&\le \sum_{i=1}^{n}\omega\big(b_i, n, (2r)^n\big) \\
&\quad\times \sum_{k\in N_1\cup N_2}\Big[K_4^2\|v_x\|^2_{L^2(Q_k\cap\Omega)} + (K_4/r)^2\|v\|^2_{L^2(Q_k\cap\Omega)} + \|u_x\|^2_{L^2(Q_k\cap\Omega)}\Big] \\
&\quad+ \sum_{i=1}^{n}\omega\big(b_i, n, (2\overline{r})^n\big) \\
&\quad\times \sum_{k\in N_3}\Big[K_5^2\|v_x\|^2_{L^2(\widehat{Q}_k\cap\Omega)} + (K_5/\overline{r})^2\|v\|^2_{L^2(\widehat{Q}_k\cap\Omega)} + \|u_x\|^2_{L^2(\widehat{Q}_k\cap\Omega)}\Big].
\end{aligned}
$$

From (44) and (24), by recalling also the connection between r and $\overline{r}$ given by (21), we easily deduce the following inequality:

$$
\begin{aligned}
(45)\quad & \Big|\sum_{i=1}^{n}\int_{\Omega} b_i u_{x_i} v\,dx\Big| \\
&\le \overline{n}K_{13}\sum_{i=1}^{n}\omega\big(b_i, n, (2\overline{r})^n\big)\Big[(1/r^2)\|v\|^2_{L^2(\Omega)} + \|u_x\|^2_{L^2(\Omega)} + \|v_x\|^2_{L^2(\Omega)}\Big],
\end{aligned}
$$

where K_{13} is a constant depending only on $K_4,\ K_5$ and therefore only on n and K. In a similar way (it is sufficient to replace b_i by d_i) we get

$$
\begin{aligned}
(46)\quad & \Big|\sum_{i=1}^{n}\int_{\Omega} d_i u v_{x_i}\,dx\Big| \\
&\le \overline{n}K_{13}\sum_{i=1}^{n}\omega(d_i, n, (2\overline{r})^n)\Big[(1/r^2)\|u\|^2_{L^2(\Omega)} + \|u_x\|^2_{L^2(\Omega)} + \|v_x\|^2_{L^2(\Omega)}\Big].
\end{aligned}
$$

Consider now the term containing the coefficient c. If $k\in N_1\cup N_2$ we have

$$
(47)\qquad \Big|\int_{\Omega\cap Q_k} cuv\,dx\Big| \le \|c\|_{L^{n/2}(\Omega\cap Q_k)}\|u\|_{L^{2^*}(\Omega\cap Q_k)}\|v\|_{L^{2^*}(\Omega\cap Q_k)}
$$

(where $2^* := 2n/(n-2)$). Taking into account Lemma 2 of [3], we get, from (47),

$$\begin{aligned}(48)\quad &\Big|\int_{\Omega\cap Q_k} cuv\,dx\Big| \\ &\le \|c\|_{L^{n/2}(\Omega\cap Q_k)} K_4^2 \left[(1/r)\|u\|_{L^2(\Omega\cap Q_k)} + \|u_x\|_{L^2(\Omega\cap Q_k)}\right] \\ &\quad \times \left[(1/r)\|v\|_{L^2(\Omega\cap Q_k)} + \|v_x\|_{L^2(\Omega\cap Q_k)}\right] \\ &\le K_4^2 \omega(c, n/2, (2\overline{r})^n) \\ &\quad \times \left[(1/r)^2 \left(\|u\|^2_{L^2(\Omega\cap Q_k)} + \|v\|^2_{L^2(\Omega\cap Q_k)}\right) + \|u_x\|^2_{L^2(\Omega\cap Q_k)} + \|v_x\|^2_{L^2(\Omega\cap Q_k)}\right].\end{aligned}$$

If, otherwise, $k \in N_3$ we have (putting again, for brevity, $\widehat{Q}_k := Q_\delta(\overline{x}_k, \overline{r})$ and recalling (23), (28)):

$$\begin{aligned}(49)\qquad &\Big|\int_{\Omega\cap Q_k} cuv\,dx\Big| \le \int_{\widehat{Q}_k\cap\Omega} |cuv|\,dx \\ &\le \omega(c, n/2, (2\overline{r})^n) K_5^2 \left[(1/\overline{r})\|u\|_{L^2(\widehat{Q}_k\cap\Omega)} + \|u_x\|_{L^2(\widehat{Q}_k\cap\Omega)}\right] \\ &\quad \times \left[(1/\overline{r})\|v\|_{L^2(\widehat{Q}_k\cap\Omega)} + \|v_x\|^2_{L^2(\widehat{Q}_k\cap\Omega)}\right].\end{aligned}$$

From (48), (49), by summing on the index k and proceeding as before (also take into account (24)) we get the following inequality:

$$\begin{aligned}(50)\qquad &\left|\int_\Omega cuv\,dx\right| \\ &\le \overline{n} K_{14} \omega(c, n/2, (2\overline{r})^n) \left[(1/r^2)\left(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\right)\right. \\ &\quad \left. + \|u_x\|^2_{L^2(\Omega)} + \|v_x\|^2_{L^2(\Omega)}\right],\end{aligned}$$

where K_{14} depends only on n, K_4, K_5 and therefore only on n and K.

It remains to consider only the last term of the bilinear form $a(.,.)$, that is the integral $\int_\Gamma guv\,d\sigma$. To this aim it is sufficient to apply (37), obtaining

$$(51)\qquad \Big|\int_\Gamma guv\,d\sigma\Big| \le K_9 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

From (41), (45), (46), (50), (51) the conclusion follows. □

Let us prove now the following result, analogous to Theorem 3.2 of [6]:

Theorem 2. *Besides the hypotheses formulated in paragraph 2, let us suppose that b_i, $d_i \in X_o^n(\Omega)$, $c \in X_o^{n/2}(\Omega)$, $g \in X_o^{n-1}(\Gamma)$ $(i = 1, 2, \ldots, n)$. Then there exists a number λ_o depending on K and the coefficients of the bilinear form $a(.,.)$ such that for any $\lambda \ge \lambda_o$ the bilinear form*

$$a(.,.) + \lambda(.,.)_{L^2(\Omega)}$$

is coercitive on $V \times V$, i.e. there exists a positive constant $\overline{K}$ such that

$$a(u,u) + \lambda\|u\|^2_{L^2(\Omega)} \ge \overline{K}\|u\|^2_{H^1(\Omega)} \quad \forall u \in V.$$

Proof. We begin from the inequalities (45), (46), (50), (51) of Theorem 1. By following a remark in paragraph 2, it is not a restriction to suppose that the number $\overline{r}$ (which exists because of the hypothesis on Γ) is sufficiently small, so that

$$2\overline{n}K_{13}\sum_{i=1}^{n}\omega\big(b_i, n, (2\overline{r})^n\big) < \nu/8 \tag{52}$$

$$2\overline{n}K_{13}\sum_{i=1}^{n}\omega\big(d_i, n, (2\overline{r})^n\big) < \nu/8 \tag{53}$$

$$2\overline{n}K_{14}\omega(c, n/2, (2\overline{r})^n) < \nu/8 \tag{54}$$

$$2\overline{n}K_{10}\omega(g, n-1, \sqrt{1+(n-1)K^2}(2\overline{r})^{n-1}) < \nu/8. \tag{55}$$

This is possible, since (as we have seen) the constants $K_{10},\ K_{13},\ K_{14}$ depend only on n and the Lipschitz constant K. From the supposed uniform ellipticity we get

$$\nu\|u_x\|^2_{L^2(\Omega)} \le \sum_{i=1}^{n}\int_\Omega a_{ij}u_{x_i}u_{x_j}\,dx \quad \forall u \in H^1(\Omega), \tag{56}$$

and from (39), (45), (46), (50), (52), (53), (54), (55)

$$\begin{aligned}&\left|\int_\Omega\Big\{\sum_{i=1}^{n}(b_i+d_i)u_{x_i}u + cu^2\Big\}dx + \int_\Gamma gu^2\,d\sigma\right| \\ &\le (\nu/2)\|u_x\|^2_{L^2(\Omega)} + (3\nu/8r^2)\|u\|^2_{L^2(\Omega)} \quad \forall u \in V.\end{aligned} \tag{57}$$

From (56), (57) the conclusion follows with $\lambda_o = (3\nu/8r^2) + \nu/2$ and $\overline{K} = \nu/2$. □

Corollary. *Suppose* $f_o \in L^{2n/(n+2)}(\Omega),\ \ f_i \in L^2(\Omega)\ (i = 1, 2, \ldots, n), h \in L^{2-2/n}(\Gamma), \lambda \ge \lambda_o$ *(see the preceding theorem). Then the boundary-value problem*

$$\begin{cases} a(u,v) + \lambda(u,v)_{L^2(\Omega)} = \displaystyle\int_\Omega\Big\{f_o v + \sum_{i=1}^{n} f_i v_{x_i}\Big\}dx + \int_\Gamma hv\,d\sigma \ \ \forall v \in V, \\ u \in V \end{cases} \tag{58}$$

has one and only one solution.

Proof. Remembering Theorem 3.3 of Stampacchia [6] it is sufficient to prove that the second member of (58) is an element of the dual of V, i.e. that there exists a constant K_{15} such that

$$\left|\int_\Omega\Big\{f_o v + \sum_{i=i}^{n} f_i v_{x_i}\Big\}dx + \int_\Gamma hv\,d\sigma\right| \le K_{15}\|v\|_{H^1(\Omega)} \qquad \forall v \in V. \tag{59}$$

First of all, from the Hölder inequality follows

$$\left|\int_\Omega f_o v\,dx\right| \le \|f_o\|_{L^{2n/(n+2)}(\Omega)}\|v\|_{L^{2n/(n-2)}(\Omega)} \tag{60}$$

$$\left|\int_\Omega \sum_{i=i}^{n} f_i v_{x_i}\,dx\right| \le \sum_{i=1}^{n}\|f_i\|_{L^2(\Omega)}\|v_{x_i}\|_{L^2(\Omega)}. \tag{61}$$

We can apply the proposition of paragraph 3, obtaining the existence of a constant K_6 such that

$$\|v\|_{L^{2n/(n-2)}(\Omega)} \le K_6\|v\|_{H^1(\Omega)} \quad \forall v \in V. \tag{62}$$

Now consider the term $\int_\Gamma hv\,d\sigma$; we have

$$\left|\int_\Gamma hv\,d\sigma\right| \le \|h\|_{L^{2-2/n}(\Gamma)}\|v\|_{L^{(2n-2)/(n-2)}(\Gamma)} \quad \forall v \in V. \tag{63}$$

Therefore the assertion will be proved as soon as we show that there exists a constant K_{16} such that

$$\|v\|_{L^{(2n-2)/(n-2)}(\Gamma)} \le K_{16}\|v\|_{H^1(\Omega)} \quad \forall v \in V, \tag{64}$$

or, more generally,

$$\|v\|_{L^{s(n-1)/(n-s)}(\Gamma)} \le K_{16}\|v\|_{H^{1,s}(\Omega)} \quad \forall v \in V,\ \text{con } 1 < s < n. \tag{65}$$

This assertion is well known since it can be deduced, for example, from [1, Theorem 5.22], or again from Theorem 5.I of Gagliardo [4] and the final remark (at the end of [4]), which extends the results to unbounded domains. All the same we report the proof for convenience of the reader and in order to explicitly calculate the constants which appear.

If we put $w := v\|v\|^{-1}_{H^{1,s}(\Omega)}$, (65) is equivalent to

$$\|w\|_{L^{s(n-1)/(n-s)}(\Gamma)} \le K_{16}\,. \tag{66}$$

First of all it turns out that

$$\begin{aligned}\int_\Gamma |w|^{s(n-1)/(n-s)}\,d\sigma &\le \sum_{k\in N_3}\int_{\Gamma\cap Q_k}|w|^{s(n-1)/(n-s)}\,d\sigma\\ &\le \sum_{k\in N_3}\int_{\Gamma\cap \widetilde{Q}_k}|w|^{s(n-1)/(n-s)}\,d\sigma,\end{aligned} \tag{67}$$

where $\widetilde{Q}_k := Q(\overline{x}_k, \overline{r})$ is one of the cubes introduced in paragraph 3. From (36) it follows that

$$\|w\|_{L^{s(n-1)/(n-s)}(\Gamma\cap\widetilde{Q}_k)} \le K_{17}\left[(1/\overline{r})\|w\|_{L^s(\Omega\cap\widetilde{Q}_k)} + \|w_x\|_{L^s(\Omega\cap\widetilde{Q}_k)}\right], \tag{68}$$

where K_{17} depends only on K and on the constants K_3 and K_4, therefore again on K and n. Recalling that $\|w\|_{H^{1,s}(\Omega)} = 1$ and that $s < s(n-1)/(n-s)$ (since $1 < s$ by hypothesis), from (68) we get

$$\|w\|^{s(n-1)/(n-s)}_{L^{s(n-1)/(n-s)}(\Gamma\cap\widetilde{Q}_k)} \leq \|w\|^{s}_{L^{s(n-1)/(n-s)}(\Gamma\cap\widetilde{Q}_k)} \leq 2^{s-1}K^s_{17}(1+1/\overline{r}^s)\|w\|^s_{H^{1,s}(\Omega\cap\widetilde{Q}_k)}. \tag{69}$$

From (69), by summing with respect to k and remembering (24), we get

$$\int_\Gamma |w|^{s(n-1)/(n-s)}\,d\sigma \leq \sum_{k\in N_3}\int_{\Gamma\cap\widetilde{Q}_k}|w|^{s(n-1)/(n-s)}\,d\sigma \tag{70}$$
$$\leq 2^{s-1}K^s_{17}(1+1/\overline{r}^s)\sum_{k\in N_3}\|w\|^s_{H^{1,s}(\Omega\cap\widetilde{Q}_k)} \leq 2^{s-1}\overline{n}K^s_{17}(1+1/\overline{r}^s)\|w\|^s_{H^{1,s}(\Omega)}$$
$$= 2^{s-1}\overline{n}K^s_{17}(1+1/\overline{r}^s),$$

so that (66) is proved, with $K_{16} = \{2^{s-1}\overline{n}K^s_{17}(1+1/\overline{r}^s)\}^{(n-s)/s(n-1)}$. □

6. A priori inequalities for subsolutions

The following result extends the theorem of [2]:

Theorem 3. *Besides the hypotheses mentioned before, suppose that:* $p > n$, $\overline{p} = p(n-1)/n$, $c \in X^{np/(n+p)}(\Omega)$, $b_i \in X^n_o(\Omega)$, $d_i \in X^p(\Omega)$, $f_i \in X^p(\Omega)$ $(i = 1,2,\dots,n)$, $f_o \in X^{np/(n+p)}(\Omega)$, $g \in X^{\overline{p}}_o(\Gamma)$, $h \in X^{\overline{p}}(\Gamma)$. *Let* $u \in H^1(\Omega)$ *such that*

$$a(u,v) \leq \int_\Omega \{f_o v + \sum_{i=1}^n f_i v_{x_i}\}\,dx + \int_\Gamma h v\,d\sigma, \tag{71}$$

for any $v \in V$, $v \geq 0$ *in* Ω *for which all the integrals in (71) make sense. Furthermore let us suppose that there exists a number* $m \geq 0$ *such that* $\max(u-m,0) \in V$.

Then there exist constants K_{18}, K_{19}, K_{20} *such that*

$$\operatorname*{ess\,sup}_\Omega u \leq K_{18}m + K_{19}\|\max(u-m,0\|_{H^1(\Omega)} + K_{20}, \tag{72}$$

where the constant K_{18} *depends only on n and p, while* K_{19}, K_{20} *depend also on the coefficients of the bilinear form* $a(.,.)$*, on the data* f_i $(i = 0,1,\dots,n)$, *h and on the regularity of the part* Γ *of the boundary of* Ω.

Proof. Let $t \geq m$ and consider the function $u_t := \max(u-t,0)$. First of all, from our hypotheses it turns out that clearly $u_t \in V$ and $u_t \geq 0$ in Ω. We want to insert u_t as v in (71) and to this aim we need to verify that, with this choice, all the integrals that appear there make sense.

Define $\Omega_t := \{x \in \Omega : u(x) > t\}$, $\Gamma_t := \{x \in \Gamma : u(x) > t\}$; we have

$$(t-m)^2\mathcal{H}_n(\Omega_t) \leq \int_\Omega u_m^2\,dx = \|u_m\|^2_{L^2(\Omega)}, \tag{73}$$

where $\mathcal{H}_n$ denotes the ordinary Lebesgue measure in $\mathbb{R}^n$. Immediately we can verify that if $t > m$ or $t \geq m > 0$, Ω_t has finite n-dimensional measure. Similarly from (64) it follows that

$$(t-m)^{\widehat{2}}\mathcal{H}_{n-1}(\Gamma_t) \leq \int_\Gamma u_m^{\widehat{2}}\,dx \leq K_{16}^{\widehat{2}}\|u_m\|_{H^1(\Omega)}^{\widehat{2}}(\Omega), \tag{74}$$

where $\mathcal{H}_{n-1}$ denotes Hausdorff $(n-1)$-dimensional measure. From (74) it clearly follows that if $t > m$ or $t \geq m > 0$ the set Γ_t has finite $(n-1)$-dimensional measure.

From all these remarks and from the hypotheses on the functions f_i $(i = 0, 1, \ldots, n)$ and h we get the reality of the integrals $\int_\Omega f_o u_t\,dx$, $\int_\Omega \sum_{i=1}^n f_i(u_t)_{x_i}\,dx$ and $\int_\Gamma h u_t\,d\sigma$, as soon as $t > m$ or $t \geq m$ with $m > 0$. Furthermore it turns out that

$$u(x) = u_t(x) + t \;\; \forall x \in \Omega_t, \qquad u_{x_i} = (u_t)_{x_i} \text{ a.e. in } \Omega_t, \tag{75}$$

whence

$$\begin{aligned} &a(u, u_t) \\ &= \int_{\Omega_t}\left\{\sum_{i=1}^n a_{ij}(u_t)_{x_i}(u_t)_{x_j} + \sum_{i=1}^n [b_i(u_t)_{x_i}u_t + d_i(u_t+t)(u_t)_{x_i}] + c(u_t+t)u_t\right\}dx \\ &\quad + \int_{\Gamma_t} g(u_t+t)\,d\sigma = a(u_t,u_t) + t\left\{\int_{\Omega_t}\left[\sum_{i=1}^n d_i(u_t)_{x_i} + cu_t\right]dx + \int_{\Gamma_t} gu_t\,d\sigma\right\}. \end{aligned} \tag{76}$$

All the integrals in (76) make sense for the following reasons. The expression $a(u_t, u_t)$ makes sense since $u_t \in V$ and because of Theorem 1. If $t = 0$ there is nothing else to prove, while if $t > 0$, as we have seen, Ω_t and Γ_t have, respectively, n-dimensional and $(n-1)$-dimensional finite measure, so that all the integrals in the last part of (76) exist. Therefore from (71), (76), taking into account the uniform ellipticity of the operator, Hölder inequality and (29), (65) we get (where, for brevity, we define $\alpha(t) := \mathcal{H}_n(\Omega_t)$, $\gamma(t) := \mathcal{H}_{n-1}(\Gamma_t)$):

$$\begin{aligned} &\nu\|u_t\|_{H^1(\Omega)}^2 \leq K_6\|f_o\|_{L^{np/(n+p)}(\Omega_t)}\|u_t\|_{H^1(\Omega)}[\alpha(t)]^{1/2-1/p} \\ &\quad + \sum_{i=1}^n \|f_i\|_{L^p(\Omega_t)}\|u_t\|_{H^1(\Omega)}[\alpha(t)]^{1/2-1/p} + K_{16}\|h\|_{L^{\overline{p}}(\Gamma_t)}\|u_t\|_{H^1(\Omega)}[\alpha(t)]^{1/2-1/p} \\ &\quad + K_6\sum_{i=1}^n\left[\|b_i\|_{L^n(\Omega_t)} + \|d_i\|_{L^n(\Omega_t)}\right]\|u_t\|_{H^1(\Omega)}^2 \\ &\quad + K_6^2\|c-\nu\|_{L^{np/(n+p)}(\Omega_t)}\|u_t\|_{H^1(\Omega)}^2[\alpha(t)]^{1/n-1/p} \\ &\quad + K_{16}^2\|g\|_{L^{\overline{p}}(\Gamma_t)}\|u_t\|_{H^1(\Omega)}^2[\alpha(t)]^{(p-n)/p(n-1)} \\ &\quad + t\left[\sum_{i=1}^n \|d_i\|_{L^p(\Omega_t)} + K_6\|c-\nu\|_{L^{np/(n+p)}(\Omega_t)} + K_{16}\|g\|_{L^{\overline{p}}(\Gamma_t)}\right] \\ &\quad \times \|u_t\|_{H^1(\Omega_t)}[\alpha(t)]^{1/2-1/p}, \end{aligned} \tag{77}$$

whence, always in a similar way to [2],

(78)

$$\begin{aligned}&\Big\{\nu - K_6 \sum_{i=1}^{n} \big[\|b_i\|_{L^n(\Omega_t)} + \|d_i\|_{L^n(\Omega_t)}\big] - K_6^2 \|c-\nu\|_{L^{np/(n+p)}(\Omega_t)} [\alpha(t)]^{1/n-1/p} \\ &\quad - K_{16}^2 \|g\|_{L^{\overline{p}}(\Gamma_t)} [\gamma(t)]^{(p-n)/p(n-1)}\Big\} \|u_t\|_{H^1(\Omega_t)} \\ &\leq \left\{K_6 \|f_o\|_{L^{np/(n+p)}(\Omega_t)} + \sum_{i=1}^{n} \|f_i\|_{L^p(\Omega_t)} + K_{16}\|h\|_{L^{\overline{p}}(\Gamma_t)}\right\} [\alpha(t)]^{(1/2-1/p)} \\ &\quad + t\left[\sum_{i=1}^{n} \|d_i\|_{L^p(\Omega_t)} + K_6 \|c-\nu\|_{L^{np/(n+p)}(\Omega_t)} + K_{16}\|g\|_{L^{\overline{p}}(\Gamma_t)}\right] [\alpha(t)]^{1/2-1/p}.\end{aligned}$$

Note that if $t \geq m$ we have

$$\int_{\Omega_m} (u-m)^2\, dx \geq \int_{\Omega_t} (u-m)^2\, dx \geq (t-m)^2 \alpha(t), \tag{79}$$

so that it turns out

$$\alpha(t) \leq \frac{\|u_m\|^2_{L^2(\Omega_m)}}{(t-m)^2} \quad \forall t > m. \tag{80}$$

Analogously, always if $t \geq m$,

$$\int_{\Gamma_m} (u-m)^{\widehat{2}}\, d\sigma \geq \int_{\Gamma_t} (u-m)^{\widehat{2}}\, d\sigma \geq (t-m)^{\widehat{2}} \gamma(t),$$

whence

$$\gamma(t) \leq \frac{\|u_m\|^{\widehat{2}}_{L^{\widehat{2}}(\Gamma_m)}}{(t-m)^{\widehat{2}}} \quad \forall t > m. \tag{81}$$

Still following [2], now we put

$$\begin{aligned}\delta_o := \min\{&1, \phi(b_i, n, \nu/(10K_6)), \phi(d_i, p, \nu/(10K_6)), \\ &\phi\big(c-\nu, np/(n+p), \nu/\big(10K_6^2\big)\big)\ (i = 1, 2, \ldots, n)\}\end{aligned} \tag{82}$$

(where ϕ is defined as in [2], formula (7), in which, as usual, we assume by definition $\inf \emptyset := +\infty$)

$$\delta_1 := \phi\big(g, \overline{p}, \nu/\big(10K_{16}^2\big)\big) \tag{83}$$

$$t_o := m + \max\left\{\|u_m\|_{L^2(\Omega_m)} \delta_o^{-1/2}, \|u_m\|_{L^{\widehat{2}}(\Gamma_m)} \delta_1^{-1/\widehat{2}}\right\}. \tag{84}$$

From (80), (81), (84) we get

$$\alpha(t_o) \le \frac{\|u_m\|^2_{L^2(\Omega_m)}}{(t_o - m)^2} \le \delta_o \tag{85}$$

$$\gamma(t_o) \le \frac{\|u_m\|^{\widehat{2}}_{L^{\widehat{2}}(\Gamma_m)}}{(t_o - m)^{\widehat{2}}}. \tag{86}$$

Taking into account also (82), (83), and Remark 2 of [2], if $t \ge t_o$ it follows that

$$\begin{cases} \sum_{i=1}^{n} \|b_i\|_{L^n(\Omega_t)} \le \nu/(10K_6) \\ \sum_{i=1}^{n} \|d_i\|_{L^p(\Omega_t)} \le \nu/(10K_6) \\ (\|c-\nu\|_{L^{np/(n+p)}(\Omega_t)} \le \nu/\big(10K_6^2\big) \\ \|g\|_{L^{\overline{p}}(\Gamma_t)} \le \nu/\big(10K_{16}^2\big), \end{cases} \tag{87}$$

so that, if $t \ge t_o$, from (78), (87) we have

$$\begin{aligned} &(3\nu/5)\|u_t\|_{H^1(\Omega_t)} \\ &\le \Bigg\{ K_6\|f_o\|_{L^{np/(n+p)}(\Omega_t)} + \sum_{i=1}^{n} \|f_i\|_{L^p(\Omega_t)} + K_{16}\|h\|_{L^{\overline{p}}(\Gamma_t)} \\ &\quad + t\Bigg[\sum_{i=1}^{n} \|d_i\|_{L^p(\Omega_t)} + K_6\|c-\nu\|_{L^{np/(n+p)}(\Omega_t)} \\ &\quad + K_{16}\|g\|_{L^{\overline{p}}(\Gamma_t)} \Bigg]\Bigg\} [\alpha(t)]^{1/2-1/p}. \end{aligned} \tag{88}$$

Inequality (88) is formally equal to (23) of [2], therefore we can, from now on, proceed in the same manner, obtaining the conclusion in the form

$$\begin{aligned} \operatorname*{ess\,sup}_{\Omega} u \le K_{18}m + K_{19}\Big[&\delta_o^{-1/2}\|u_m\|_{L^2(\Omega)} + \delta_1^{-1/\widehat{2}}\|u_m\|_{H^1(\Omega)} \\ &+ \omega(f_o, np/(n+p), \delta_o) + \sum_{i=1}^{n} \omega(f_i, p, \delta_o) + \omega(h, \overline{p}, \delta_1)\Big], \end{aligned} \tag{89}$$

where K_{18} is a constant depending only on n, p, while K_{19} depends also on K_6 and K_{16}. Note that if $g = 0$ it follows that $\delta_1 = +\infty$ so that in this case the term $\|u_m\|_{H^1(\Omega)}$ does not appear in the second member (as already happened in [2]). □

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