A PRIORI INEQUALITIES IN $L^\infty(\Omega)$ FOR SOLUTIONS OF ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. We prove some a priori inequalities in $L^\infty(\Omega)$ for subsolutions of elliptic equations in divergence form, with Dirichlet's boundary conditions, in unbounded domains.

1. Introduction.

In an open subset Ω of $\mathbb{R}^n$, not necessarily bounded, we consider a linear uniformly elliptic second order operator in variational form with discontinuous coefficients, associated to the bilinear form

$$a(u,v)=\int_\Omega\{\sum_{i,j=1}^n a_{ij}u_{x_i}v_{x_j}+\sum_{i=1}^n(b_iu_{x_i}v+d_iuv_{x_i})+cuv\}\,dx \tag{1}$$

If one supposes that $u\in H^1(\Omega)$ is a solution of the inequality

$$a(u,v)\le\int_\Omega\{f_ov+\sum_{i=1}^n f_iv_{x_i}\}\,dx\quad\forall\ v\in C_o^1(\Omega),\ v\ge 0 \text{ in } \Omega \tag{2}$$

one can consider the problem of determining the minimal hypotheses on the coefficients $b_i,\ d_i,\ c$ of the bilinear form (1) and on the known functions f_i $(i=0,1,...,n)$ in order that the subsolution u is (essentially) bounded from above in Ω. Such a problem was already studied e. g. in [2] and [3], where an inequality of the kind

$$\text{ess sup}_\Omega u\le\max(0,\max_{\partial\Omega}u)+K_1\{||f_o||_{L^{p/2}(\Omega)}+\sum_{i=1}^n||f_i||_{L^p(\Omega)}\}+K_2||u||_{L^2(\Omega)} \tag{3}$$

was proved, by supposing Ω bounded and $f_i,\ d_i\in L^p(\Omega),\ f_o,\ c\in L^{p/2}(\Omega),\ p>n.$

The aim of the present work is to extend these results by permitting first of all that the set Ω is unbounded and by making more general hypotheses on the functions $f_o,\ f_i,\ b_i,\ d_i,\ c$ $(i=1,2...,n)$. Finally, the constants in the a priori inequality (3) are explicitly evaluated, differently from what happened in the previous works [2] and [3].

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2. Notations and Hypotheses.

Let Ω be an open subset (bounded or unbounded) of $\mathbb{R}^n$. Let $a_{ij} \in L^\infty(\Omega)$ $(i,j=1,2,...,n)$, $\sum_{i,j=1}^n a_{ij}t_i t_j \geq \nu|t|^2\ \forall t \in \mathbb{R}^n$ a.e. in Ω, where ν is a positive constant. Let $c^+ := \max(c,0)$, $c^- := \min(c,0)$ and suppose that $c^+ \in L^{2n/(n+2)}(\Omega')$ for any Ω' bounded, $\Omega' \subset \Omega$. Let us define the spaces

$$X^p(\Omega) := \{f \in L^p_{loc}(\Omega) :\ \omega(f,p,\delta) < +\infty\ \forall \delta > 0\} \tag{4}$$

$$X^p_o(\Omega) := \{f \in X^p(\Omega) :\ \lim_{\delta \to 0^+} \omega(f,p,\delta) = 0\} \tag{5}$$

where

$$\omega(f,p,\delta) := \sup\{||f||_{L^p(E)} :\ E \text{ measurable},\ E \subset \Omega,\ \text{meas}E \leq \delta\} \tag{6}$$

Remark 1. If $f \in L^p_{loc}(\Omega)$ and we define, for $k > 0$,

$$\phi(f,p,k) := \inf\{\text{meas}E :\ E \text{ measurable},\ E \subset \Omega,\ ||f||_{L^p(E)} \geq k\} \tag{7}$$

we have

$$f \in X^p(\Omega) \quad \text{if and only if} \quad \exists\ k_o > 0 \text{ such that } \phi(f,p,k_o) > 0 \tag{8}$$

$$f \in X^p_o(\Omega) \quad \text{if and only if} \quad \phi(f,p,k) > 0 \quad \forall k > 0 \tag{9}$$

Remark 2. If G is a measurable subset of Ω such that meas $G \leq \phi(f,p,k)$, then it turns out $||f||_{L^p(G)} \leq k$. In fact, if not there would exist a subset G_o of G with measure positive but so small that

$$||f||_{L^p(G \setminus G_o)} > k$$

which is contrary to the definition of ϕ, since $\text{meas}(G \setminus G_o) < \text{meas}G$. $\square$

Remark 3. If $1 \leq q < p$ it turns out $X^p(\Omega) \subset X^q_o(\Omega)$.

In fact, if $E \subset \Omega$, $\text{meas}E \leq \delta$, $f \in X^p(\Omega)$ we have

$$||f||_{L^q(E)} \leq ||f||_{L^p(E)} (\text{meas}E)^{(p-q)/pq} \leq \omega(f,p,\delta)\delta^{(p-q)/pq}$$

whence

$$\omega(f,q,\delta) \leq \omega(f,p,\delta)\delta^{(p-q)/pq}\ \square$$

We denote by S the constant in the Sobolev inequality

$$||g||_{L^{2n/(n-2)}(\mathbb{R}^n)} \leq S||g_x||_{L^2(\mathbb{R}^n)}\ \forall g \in C^1_o(\mathbb{R}^n)$$

As it is well known (see e.g. [4]), it is

$$S = [n(n-2)\pi]^{-1/2}\Gamma(n)^{1/n}\Gamma(n/2)^{-1/n} \tag{10}$$

3. Main Result.

Theorem. *Besides the hypotheses mentioned above, let us suppose* $c^- \in X_o^{pn/(n+p))}(\Omega), b_i \in X_o^n(\Omega),\ d_i \in X_o^p(\Omega),\ f_i \in X^p(\Omega)\ (i = 1, 2, ..., n)$, $f_o \in X^{np/(n+p)}(\Omega), u \in H^1_{loc}(\Omega),\ p > n,$

$$a(u, v) \leq \int_\Omega \{f_o v + \sum_{i=1}^n f_i v_{x_i}\}\, dx \quad \forall v \in C^1_o(\Omega),\ v \geq 0 \text{ in } \Omega. \tag{11}$$

Let us suppose furthermore that there exists a real nonnegative number m such that $\max(u - m, 0) \in H^1_o(\Omega)$.

Then there exist constants $K_1,\ K_2,\ K_3$, depending on the coefficients of $a(.,.)$, on n and p, such that

$$\begin{aligned} &\text{ess sup}_\Omega u \leq K_1 || \max(u - m, 0)||_{L^2(\Omega)} + 2^{np/(p-n)} m + \\ &\qquad + K_2\{S\omega(f_o, np/(p + n), K_3) + \sum_{i=1}^n \omega(f_i, p, K_3)\} \end{aligned} \tag{12}$$

where:
S is the Sobolev constant (10),
$K_1 = (4/3)^{np/(p-n)} + 2^{np/(p-n)} K_3^{-1/2}$,
$K_2 = (3S/\nu)[2^{np/(p-n)} - 1]$,
$K_3 = \min\{1, \phi(b_i, n, \nu/(6Sn)), \phi(d_i, p, \nu/(6Sn)), \phi(c^-, np/(p + n), \nu/(6S^2))$
$(i = 1, 2, ..., n)\}$

Proof. Let us remark, first of all, that if $t \geq m$ obviously it is also $u_t := \max(u - t, 0) \in H^1_o(\Omega)$. It is also easy to verify that (11) is fulfilled also by nonnegative functions $v \in H^1_o(\Omega)$ with bounded support. Therefore it is possible to put in (11) $v = u_t$ provided $t \geq m$. Let us denote for brevity

$$\Omega_t := \{x \in \Omega :\ u(x) > t\}$$

By using Hölder's and Sobolev's inequalities, and taking into account our previous hypotheses, we deduce

$$\nu||(u_t)_x||^2_{L^2(\Omega_t)} \leq \int_{\Omega_t} a_{ij} u_{x_i} (u_t)_{x_j}\, dx,$$

$$\left|\int_\Omega \sum_{i=1}^n b_i u_{x_i} u_t\, dx\right| \leq \sum_{i=1}^n \int_{\Omega_t} |b_i (u_t)_{x_i} u_t|\, dx \leq S \sum_{i=1}^n ||b_i||_{L_n(\Omega_t)} ||(u_t)_x||^2_{L^2(\Omega_t)},$$

$$\begin{aligned} \left|\int_\Omega \sum_{i=1}^n d_i u (u_t)_{x_i}\, dx\right| &\leq \sum_{i=1}^n \int_{\Omega_t} |d_i u_t (u_t)_{x_i}|\, dx + t \sum_{i=1}^n \int_{\Omega_t} |d_i (u_t)_{x_i}|\, dx \leq \\ &\leq S \sum_{i=1}^n ||d_i||_{L^p(\Omega_t)} (\text{meas}\Omega_t)^{(p-n)/np} ||(u_t)_x||^2_{L^2(\Omega_t)} + \end{aligned}$$

$$+\ t \sum_{i=1}^{n} ||d_i||_{L^p(\Omega_t)} (\mathrm{meas}\Omega_t)^{(p-2)/2p} ||(u_t)_x||_{L^2(\Omega_t)},$$

$$|\int_{\Omega} c^- u u_t \, dx| \leq \int_{\Omega_t} |c^- u_t^2| \, dx + t \int_{\Omega_t} |c^- u_t| \, dx \leq$$

$$\leq S^2 ||c^-||_{L^{np/(n+p)}(\Omega_t)} (\mathrm{meas}\Omega_t)^{(p-n)/np} ||(u_t)_x||^2_{L^2(\Omega_t)} +$$

$$+\ tS ||c^-||_{L^{np/(p-n)}(\Omega_t)} (\mathrm{meas}\Omega_t)^{(p-2)/2p} ||(u_t)_x||_{L^2(\Omega_t)},$$

$$|\int_{\Omega} f_o u_t \, dx| \leq S ||f_o||_{L^{np/(n+p)}(\Omega_t)} (\mathrm{meas}\Omega_t)^{(p-2)/2p} ||(u_t)_x||_{L^2(\Omega_t)},$$

$$|\int_{\Omega} \sum_{i=1}^{n} f_i (u_t)_{x_i} \, dx| \leq \sum_{i=1}^{n} ||f_i||_{L^p(\Omega_t)} (\mathrm{meas}\Omega_t)^{(p-2)/2p} ||(u_t)_x||_{L^2(\Omega_t)}.$$

Therefore from (11) it follows easily

$$\nu ||(u_t)_x||^2_{L^2(\Omega_t)} \leq \tag{13}$$

$$\leq t[\textstyle\sum_{i=1}^{n} ||d_i||_{L^p(\Omega_t)} + S||c^-||_{L^{np/(n+p)}(\Omega_t)}](\mathrm{meas}\Omega_t)^{(p-2)/2p} ||(u_t)_x||_{L^2(\Omega_t)} +$$

$$+\ [S||f_o||_{L^{np/(n+p)}(\Omega_t)} + \textstyle\sum_{i=1}^{n} ||f_i||_{L^p(\Omega_t)}](\mathrm{meas}\Omega_t)^{(p-2)/2p} ||(u_t)_x||_{L^2(\Omega_t)} +$$

$$+\ S\left[\textstyle\sum_{i=1}^{n} ||b_i||_{L^n(\Omega_t)} + \sum_{i=1}^{n} ||d_i||_{L^p(\Omega_t)} (\mathrm{meas}\Omega_t)^{(p-n)/np}\right] ||(u_t)_x||^2_{L^2(\Omega_t)} +$$

$$+ S^2 ||c^-||_{L^{np/(n+p)}(\Omega_t)} (\mathrm{meas}\Omega_t)^{(p-n)/np} ||(u_t)_x||^2_{L^2(\Omega_t)}$$

By putting, for brevity, $\alpha(t) := \mathrm{meas}\Omega_t$, we get

$$\begin{aligned}
&\Bigg\{\nu - S\Bigg[\sum_{i=1}^{n} ||b_i||_{L^n(\Omega_t)} + \sum_{i=1}^{n} ||d_i||_{L^p(\Omega_t)} [\alpha(t)]^{(p-n)/np} + \\
&\quad + S||c^-||_{L^{np/(n+p)}(\Omega_t)} [\alpha(t)]^{(p-n)/np}\Bigg]\Bigg\} ||(u_t)_x||_{L^2(\Omega_t)} \leq \\
&\leq \Bigg[S||f_o||_{L^{np/(n+p)}(\Omega_t)} + \sum_{i=1}^{n} ||f_i||_{L^p(\Omega_t)}\Bigg] [\alpha(t)]^{(p-2)/2p} + \\
&\quad + t\Bigg[\sum_{i=1}^{n} ||d_i||_{L^p(\Omega_t)} + S||c^-||_{L^{np/(n+p)}(\Omega_t)}\Bigg] [\alpha(t)]^{(p-2)/2p}
\end{aligned} \tag{14}$$

Let us remark that, when $t \geq m$, we have

$$\int_{\Omega_m} (u-m)^2 \, dx \geq \int_{\Omega_t} (u-m)^2 \, dx \geq (t-m)^2 \alpha(t)$$

i. e.

$$\alpha(t) \leq \frac{||u_m||^2_{L_2(\Omega_m)}}{(t-m)^2} \quad \forall t > m \tag{15}$$

Now we define (see (7))

$$\delta_o := \min\left\{1, \phi\left(b_i, n, \nu/(6Sn)\right), \phi\left(d_i, p, \nu/(6Sn)\right), \phi\left(c^-, np/(n+p), \nu/(6S^2)\right), (i = 1, 2, ..., n)\right\} \tag{16}$$

$$t_o := m + \frac{||u_m||_{L^2(\Omega)}}{\delta_o^{1/2}} \tag{17}$$

(please note that $\delta_o > 0$ because of our previous hypotheses and remark 1).
Then if $t \geq t_o$ we have

$$\alpha(t) \leq \alpha(t_o) \leq \frac{||u_m||^2_{L^2(\Omega)}}{(t_o - m)^2} = \delta_o \tag{18}$$

therefore by the definition of ϕ and remark 2 we deduce

$$\begin{cases} \sum_{i=1}^n ||b_i||_{L^n(\Omega_t)} \leq \nu/(6S), \quad \sum_{i=1}^n ||d_i||_{L^p(\Omega_t)} \leq \nu/(6S), \\ ||c^-||_{L^{np/(n+p)}(\Omega_t)} \leq \nu/(6S^2) \end{cases} \tag{19}$$

From (15), (16), (18) it follows $\alpha(t) \leq 1$, then from (14), (19) when $t \geq t_o$ we get

$$(\nu/2)||(u_t)_x||_{L^2(\Omega_t)} \leq \leq [\alpha(t)]^{(p-2)/2p}\left[t\left(\sum_{i=1}^n ||d_i||_{L^p(\Omega_t)} + S||c^-||_{L^{np/(n+p)}(\Omega_t)}\right) + +S||f_o||_{L^{np/(n+p)}(\Omega_t)} + \sum_{i=1}^n ||f_i||_{L^p(\Omega_t)}\right] \tag{20}$$

Let us denote, for brevity,

$$\begin{cases} K_4 := (2S/\nu)\left(\sum_{i=1}^n ||d_i||_{L^p(\Omega_{t_o})} + S||c^-||_{L^{np/(n+p)}(\Omega_{t_o})}\right) \\ K_5 := (2S/\nu)\left(\sum_{i=1}^n ||f_i||_{L^p(\Omega_{t_o})} + S||f_o||_{L^{np/(n+p)}(\Omega_{t_o})}\right) \end{cases} \tag{21}$$

and apply to (20) Hölder's and Sobolev's inequalitites, thus obtaining

$$||u_t||_{L^1(\Omega_t)} \leq [\alpha(t)]^{(2+n)/2n}||u_t||_{L^{2n/(n-2)}(\Omega_t)} \leq [\alpha(t)]^{1+(p-n)/np}(K_4 t + K_5) \tag{22}$$

Now we follow a procedure of [1]. Put, for $t \geq t_o$,

$$\beta(t) := ||u_t||_{L^1(\Omega_t)} \tag{23}$$

and note that it turns out $\beta(t) = \int_t^{+\infty} \alpha(s)\, ds$, therefore

$$\beta'(t) = -\alpha(t) \leq 0 \quad \text{a.e. in } [t_o, +\infty) \tag{24}$$

From (22), (24) we deduce the differential inequality

$$\beta(t) \le (K_4 t + K_5)[-\beta'(t)]^{1+(p-n)/np} \quad \text{a.e. in } [t_o, +\infty) \tag{25}$$

Suppose now, by contradiction, that $\beta(t) > 0 \quad \forall t \ge t_o$ (i.e., by definition of $\beta(t)$, ess $\sup_\Omega u = +\infty$). Then in (25) we can divide by $\beta(t)$ thus obtaining

$$-\beta'(t)[\beta(t)]^{-np/(np+p-n)} \ge (K_4 t + K_5)^{-np/(np+p-n)} \tag{26}$$

By integrating (26) from t_o and $t^* > t_o$ (suppose for the moment $K_4 > 0$), we obtain

$$K_4[\beta(t_o)]^{(p-n)/(np+p-n)} - K_4[\beta(t^*)]^{(p-n)/(np+p-n)} \ge \tag{27}$$

$$\ge (K_4 t^* + K_5)^{(p-n)/(np+p-n)} - (K_4 t_o + K_5)^{(p-n)/(np+p-n)}$$

In (27) we get a contradiction when t^* tends to $+\infty$, so it follows ess $\sup_\Omega u < +\infty$. We can write again (27) with $t_o < t^* <$ ess $\sup_\Omega u$; by letting t^* tend to ess $\sup_\Omega u$ we deduce

$$(K_4 \text{ess} \sup_\Omega u + K_5)^{(p-n)/(np+p-n)} \le \tag{28}$$

$$\le (K_4 t_o + K_5)^{(p-n)/(np+p-n)} + K_4[\beta(t_o)]^{(p-n)/(np+p-n)}$$

Please note that the constant K_4 is not greater than 2/3 because of (19). From (28) by easy calculations we get

$$\text{ess} \sup_\Omega u \le (4/3)^{np/(p-n)}||u_{t_o}||_{L^1(\Omega_{t_o})} + 2^{np/(p-n)} t_o + (3/2)[2^{np/(p-n)} - 1]K_5 \tag{29}$$

whence, by recalling the definition of t_o (given by (17)) and K_5 (given by (21)) one can write

$$\text{ess} \sup_\Omega u \le 2^{np/(p-n)} m + [(4/3)^{np/(p-n)} + 2^{np/(p-n)} \delta_o^{-1/2}]||u_m||_{L^2(\Omega)} + \tag{30}$$

$$+(3S/\nu)[2^{np/(p-n)-1]} \left[S||f_o||_{L^{np/(p+n)}(\Omega_{t_o})} + \sum_{i=1}^n ||f_i||_{L^p(\Omega_{t_o})} \right]$$

Finally, by taking into account (18), the definition of δ_o (see (16)) and the functions ϕ, ω, we conclude

$$\text{ess} \sup_\Omega u \le 2^{np/(p-n)} m + [(4/3)^{np/(p-n)} + 2^{np/(p-n)} \delta_o^{-1/2}]||u_m||_{L^2(\Omega)} + \tag{31}$$

$$+ (3S/\nu)[2^{np/(p-n)} - 1] \left[S\omega(f_o, np/(p+n), \delta_o) + \sum_{i=1}^n \omega(f_i, p, \delta_o) \right]$$

with δ_o given by (16). $\square$

Remark 4. If we suppose, besides the hypotheses of the preceding theorem, that there exists $q \geq 1$ such that $u_m \in L^q(\Omega)$, then we can write, instead of (15), (17)

$$\alpha(t) \leq ||u_m||^q_{L^q(\Omega_m)}(t-m)^{-q} \quad \forall t > m, \tag{15'}$$

$$t_o := m + ||u_m||_{L^q(\Omega)}\delta_o^{-1/q} \tag{17'}$$

and by proceeding as before we arrive at the conclusion in the form

$$\text{ess sup}_\Omega u \leq 2^{np/(p-n)}m + [(4/3)^{np/(p-n)} + 2^{np/(p-n)}\delta_o^{-1/q}]||u_m||_{L^q(\Omega)} + \tag{31'}$$

$$+ (3S/\nu)[2^{np/(p-n)-1}\big[S\omega(f_o, np/(p+n), \delta_o) + \sum_{i=1}^{n} \omega(f_i, p, \delta_o)\big]$$

where δ_o is always given by (16).

Remark 5. Suppose, in the bilinear form $a(.,.)$, that the coefficients d_i and c^- be identically zero. Then the constant K_4 defined by the first of (21) vanishes, and by integrating (25) we get, more simply,

$$\text{ess sup}_\Omega u \leq t_o + (np+p-n)/(p-n)\, K_5^{np/(np+p-n)}||u_{t_o}||^{(p-n)/(np+p-n)}_{L^2(\Omega)} \tag{32}$$

whence, by taking into account the definitions of t_o, $\delta_o, ...$, and Young's inequality, we deduce

$$\text{ess sup}_\Omega u \leq m + (\delta_o^{-1/2} + 1)||u_m||_{L^2(\Omega)} + [np/(p-n)]K_5 \tag{33}$$

This inequality is of the same kind of (31), but the coefficient of m in it is now 1.

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