A PRIORI INEQUALITIES IN $L^\infty(\Omega)$ FOR SOLUTIONS OF ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. We prove some a priori inequalities in $L^\infty(\Omega)$ for subsolutions of elliptic equations in divergence form, with Dirichlet’s boundary conditions, in unbounded domains.

1. Introduction.

In an open subset $\Omega$ of $\mathbb{R}^n$, not necessarily bounded, we consider a linear uniformly elliptic second order operator in variational form with discontinuous coefficients, associated to the bilinear form

\[ a(u, v) = \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n (b_i u_{x_i} v + d_i u v_{x_i}) + c u v \right\} \, dx \]

If one supposes that $u \in H^1(\Omega)$ is a solution of the inequality

\[ a(u, v) \leq \int_\Omega \left\{ f_0 v + \sum_{i=1}^n f_i v_{x_i} \right\} \, dx \quad \forall \, v \in C^1_c(\Omega), \, v \geq 0 \text{ in } \Omega \]

one can consider the problem of determining the minimal hypotheses on the coefficients $b_i, \, d_i, \, c$ of the bilinear form (1) and on the known functions $f_i \ (i = 0, 1, ..., n)$ in order that the subsolution $u$ is (essentially) bounded from above in $\Omega$. Such a problem was already studied e. g. in [2] and [3], where an inequality of the kind

\[ \text{ess sup}_{\Omega} u \leq \max(0, \max_{\partial \Omega} u) + K_1 \left\{ \|f_0\|_{L^{p/2}(\Omega)} + \sum_{i=1}^n \|f_i\|_{L^p(\Omega)} \right\} + K_2 \|u\|_{L^2(\Omega)} \]

was proved, by supposing $\Omega$ bounded and $f_i, \, d_i \in L^p(\Omega), \, f_0, \, c \in L^{p/2}(\Omega), \, p > n$.

The aim of the present work is to extend these results by permitting first of all that the set $\Omega$ is unbounded and by making more general hypotheses on the functions $f_o, \, f_i, \, b_i, \, d_i, \, c \ (i = 1, 2, ..., n)$. Finally, the constants in the a priori inequality (3) are explicitly evaluated, differently from what happened in the previous works [2] and [3].
2. Notations and Hypotheses.

Let \( \Omega \) be an open subset (bounded or unbounded) of \( \mathbb{R}^n \). Let \( a_{ij} \in L^\infty(\Omega) \), \( i, j = 1, 2, \ldots, n \), \( \sum_{i,j=1}^n a_{ij} t_i t_j \geq \nu |t|^2 \forall t \in \mathbb{R}^n \) a.e. in \( \Omega \), where \( \nu \) is a positive constant. Let \( c^+ \):= \max(c, 0), \( c^- := \min(c, 0) \) and suppose that \( c^+ \in L^{2n/(n+2)}(\Omega') \) for any \( \Omega' \) bounded, \( \Omega' \subset \Omega \). Let us define the spaces

\[ X^p(\Omega) := \{ f \in L^p_{\text{loc}}(\Omega) : \omega(f, p, \delta) < +\infty \forall \delta > 0 \} \]

\[ X^p_0(\Omega) := \{ f \in X^p(\Omega) : \lim_{\delta \to 0^+} \omega(f, p, \delta) = 0 \} \]

where

\[ \omega(f, p, \delta) := \sup \{ ||f||_{L^p(E)} : E \text{ measurable}, E \subset \Omega, \text{meas} E \leq \delta \} \]

Remark 1. If \( f \in L^p_{\text{loc}}(\Omega) \) and we define, for \( k > 0 \),

\[ \phi(f, p, k) := \inf \{ \text{meas} E : E \subset \Omega, ||f||_{L^p(E)} \geq k \} \]

we have

\[ f \in X^p(\Omega) \text{ if and only if } \exists k_0 > 0 \text{ such that } \phi(f, p, k_0) > 0 \]

\[ f \in X^p_0(\Omega) \text{ if and only if } \phi(f, p, k) > 0 \ \forall k > 0 \]

Remark 2. If \( G \) is a measurable subset of \( \Omega \) such that \( \text{meas} G \leq \phi(f, p, k) \), then it turns out \( ||f||_{L^p(G)} \leq k \). In fact, if not there would exist a subset \( G_0 \) of \( G \) with measure positive but so small that

\[ ||f||_{L^p(G \setminus G_0)} > k \]

which is contrary to the definition of \( \phi \), since \( \text{meas}(G \setminus G_0) < \text{meas} G \). \( \square \)

Remark 3. If \( 1 \leq q < p \) it turns out \( X^p(\Omega) \subset X^q_0(\Omega) \).

In fact, if \( E \subset \Omega, \text{meas} E \leq \delta, f \in X^p(\Omega) \) we have

\[ ||f||_{L^q(E)} \leq ||f||_{L^p(E)}(\text{meas} E)^{(p-q)/pq} \leq \omega(f, p, \delta)^{(p-q)/pq} \delta^{(p-q)/pq} \]

whence

\[ \omega(f, q, \delta) \leq \omega(f, p, \delta)^{p-q}/pq \ \square \]

We denote by \( S \) the constant in the Sobolev inequality

\[ ||g||_{L^{2n/(n-2)}(\mathbb{R}^n)} \leq S ||g_x||_{L^2(\mathbb{R}^n)} \ \forall g \in C^1_0(\mathbb{R}^n) \]

As it is well known (see e.g. [4]), it is

\[ S = \frac{n(n-2)\pi^{-1/2} \Gamma(n/2) \Gamma(1/2)^{-1/n}}{n(2n-2)\pi^{-1/2} \Gamma(n/2)^{1/n}} \]
3. Main Result.

Theorem. Besides the hypotheses mentioned above, let us suppose $c^{-} \in X_{o}^{p,\nu/(n+p)}(\Omega), b_{i} \in X_{o}^{p}(\Omega), d_{i} \in X_{o}^{p}(\Omega), f_{i} \in X^{p}(\Omega)$ ($i = 1, 2, \ldots, n$), $f_{0} \in X^{np/(n+p)}(\Omega), u \in H_{0}^{1}(\Omega)$, $p > n$,

\begin{equation}
(11) \quad a(u, v) \leq \int_{\Omega} \left\{ f_{0}v + \sum_{i=1}^{n} f_{i}v_{x_{i}} \right\} dx \quad \forall v \in C_{o}^{1}(\Omega), \ v \geq 0 \text{ in } \Omega.
\end{equation}

Let us suppose furthermore that there exists a real nonnegative number $m$ such that $\max(u - m, 0) \in H_{0}^{1}(\Omega)$.

Then there exist constants $K_{1}$, $K_{2}$, $K_{3}$, depending on the coefficients of $a(\cdot, \cdot)$, on $n$ and $p$, such that

\begin{equation}
(12) \quad \text{ess sup}_{\Omega} u \leq K_{1} \left| \max(u - m, 0) \right|_{L^{2}(\Omega)} + 2^{np/(p-n)} m + K_{2} \left\{ S \omega(f_{0}, np/(p+n), K_{3}) + \sum_{i=1}^{n} \omega(f_{i}, p, K_{3}) \right\},
\end{equation}

where:

$S$ is the Sobolev constant (10),

$K_{1} = (4/3)^{np/(p-n)} + 2^{np/(p-n)} K_{3}^{-1/2},$

$K_{2} = (3S/\nu)[2^{np/(p-n)} - 1],$

$K_{3} = \min \{ 1, \phi(b, n, \nu/(6Sn)), \phi(d, p, \nu/(6Sn)), \phi(c^{-}, np/(p+n), \nu/(6S^{2})) \} (i = 1, 2, \ldots, n)$.\]

Proof. Let us remark, first of all, that if $t \geq m$ obviously it is also $u_{t} := \max(u - t, 0) \in H_{0}^{1}(\Omega)$. It is also easy to verify that (11) is fulfilled also by nonnegative functions $v \in H_{0}^{1}(\Omega)$ with bounded support. Therefore it is possible to put in (11) $v = u_{t}$ provided $t \geq m$. Let us denote for brevity

\[ \Omega_{t} := \{ x \in \Omega : u(x) > t \} \]

By using Hölder’s and Sobolev’s inequalities, and taking into account our previous hypotheses, we deduce

\[ \nu \| (u_{t})_{x} \|_{L^{2}(\Omega_{t})}^{2} \leq \int_{\Omega_{t}} a_{ij} u_{x_{i}} (u_{t})_{x_{j}} dx, \]

\[ \left| \int_{\Omega_{t}} \sum_{i=1}^{n} b_{i} u_{x_{i}} u_{t} dx \right| \leq \sum_{i=1}^{n} \int_{\Omega_{t}} |b_{i} u_{x_{i}} u_{t}| dx \leq S \sum_{i=1}^{n} \| b_{i} \|_{L^{n}(\Omega_{t})} \| (u_{t})_{x} \|_{L^{2}(\Omega_{t})}^{2}, \]

\[ \left| \int_{\Omega_{t}} \sum_{i=1}^{n} d_{i} u (u_{t})_{x_{i}} dx \right| \leq \sum_{i=1}^{n} \int_{\Omega_{t}} |d_{i} u (u_{t})_{x_{i}}| dx + t \sum_{i=1}^{n} \int_{\Omega_{t}} |(u_{t})_{x_{i}}| dx \leq \]

\[ \leq S \sum_{i=1}^{n} \| d_{i} \|_{L^{p}(\Omega_{t})} \| \text{meas} \Omega_{t} \|^{(p-n)/np} \| (u_{t})_{x} \|_{L^{2}(\Omega_{t})} + \]

\[ + t \sum_{i=1}^{n} |d_i|_{L^p(\Omega_2)} (\text{meas} \Omega_2)^{(p-2)/2p} \| (u_t)_x \|_{L^2(\Omega_2)}, \]

\[ | \int_{\Omega} c^- u_t \, dx | \leq \int_{\Omega} |c^- u_t^2| \, dx + t \int_{\Omega} |c^- u_t| \, dx \leq \]

\[ \leq S^2 \| c^- \|_{L^{n/p}(\Omega_2)} (\text{meas} \Omega_2)^{(p-n)/np} \| (u_t)_x \|_{L^2(\Omega_2)}^2 + \]

\[ + tS \| c^- \|_{L^{n/p}(\Omega_2)} (\text{meas} \Omega_2)^{(p/2-2)/2p} \| (u_t)_x \|_{L^2(\Omega_2)}, \]

Therefore from (11) it follows easily

(13) \[ \nu \| (u_t)_x \|_{L^2(\Omega_1)}^2 \]

\[ \leq t \sum_{i=1}^{n} |d_i|_{L^p(\Omega_1)} + S \| c^- \|_{L^{n/p}(\Omega_2)} (\text{meas} \Omega_2)^{(p-2)/2p} \| (u_t)_x \|_{L^2(\Omega_2)} + \]

\[ + [S \| f_0 \|_{L^{n/p}(\Omega_2)} + \sum_{i=1}^{n} |d_i|_{L^p(\Omega_1)} (\text{meas} \Omega_2)^{(p-2)/2p} \| (u_t)_x \|_{L^2(\Omega_2)} + \]

\[ + S \| \sum_{i=1}^{n} b_i \|_{L^p(\Omega_1)} + \sum_{i=1}^{n} |d_i|_{L^p(\Omega_1)} (\text{meas} \Omega_2)^{(p-n)/np} \| (u_t)_x \|_{L^2(\Omega_2)}^2 + \]

\[ + S^2 \| c^- \|_{L^{n/p}(\Omega_2)} (\text{meas} \Omega_2)^{(p-n)/np} \| (u_t)_x \|_{L^2(\Omega_2)}^2 \]

By putting, for brevity, \( \alpha(t) := \text{meas} \Omega_2 \), we get

(14) \[ \{ \nu - S \left[ \sum_{i=1}^{n} |b_i|_{L^p(\Omega_1)} + \sum_{i=1}^{n} |d_i|_{L^p(\Omega_1)} |\alpha(t)|^{(p-n)/np}\right] + \]

\[ + S \| c^- \|_{L^{n/p}(\Omega_2)} |\alpha(t)|^{(p-n)/np} \} \| (u_t)_x \|_{L^2(\Omega_2)} \leq \]

\[ \leq \left[ S \| f_0 \|_{L^{n/p}(\Omega_2)} + \sum_{i=1}^{n} |f_i|_{L^p(\Omega_1)} \right] |\alpha(t)|^{(p-2)/2p} + \]

\[ + t \left[ \sum_{i=1}^{n} |d_i|_{L^p(\Omega_1)} + S \| c^- \|_{L^{n/p}(\Omega_2)} \right] |\alpha(t)|^{(p-2)/2p} \]

Let us remark that, when \( t \geq m \), we have

\[ \int_{\Omega_m} (u - m)^2 \, dx \geq \int_{\Omega_t} (u - m)^2 \, dx \geq (t - m)^2 \alpha(t) \]

i. e.

(15) \[ \alpha(t) \leq \frac{||u_m||_{L^2(\Omega_m)}^2}{(t - m)^2} \quad \forall t > m \]
Now we define (see (7))

\[ \delta_o := \min \{ 1, \phi(b_i, n, \nu/(6Sn)), \phi(d_i, p, \nu/(6Sn)), \phi(e^-, np/(n + p), \nu/(6S^2)) \}, (i = 1, 2, \ldots, n) \]  

\[ t_o := m + \frac{||u_m||_{L^2(\Omega)}}{\delta_o^{1/2}} \]

(please note that \( \delta_o > 0 \) because of our previous hypotheses and remark 1).

Then if \( t \geq t_o \) we have

\[ \alpha(t) \leq \alpha(t_o) \leq ||u_m||_{L^2(\Omega)} \frac{1}{(t_o - m)^2} = \delta_o \]

therefore by the definition of \( \phi \) and remark 2 we deduce

\[ \sum_{i=1}^n ||b_i||_{L^n(\Omega_i)} \leq \nu/(6S), \quad \sum_{i=1}^n ||d_i||_{L^p(\Omega_i)} \leq \nu/(6S), \quad ||e^-||_{L^{np/(n+p)}(\Omega_i)} \leq \nu/(6S^2) \]

From (15), (16), (18) it follows \( \alpha(t) \leq 1 \), then from (14), (19) when \( t \geq t_o \) we get

\[ (\nu/2)||u_t||_{L^2(\Omega)} \leq \]

\[ \leq [\alpha(t)]^{(p-2)/2p} \left[ t \left( \sum_{i=1}^n ||d_i||_{L^p(\Omega_i)} + S||e^-||_{L^{np/(n+p)}(\Omega_i)} \right) + \right. \]

\[ \left. + S||f_0||_{L^{np/(n+p)}(\Omega_i)} + \sum_{i=1}^n ||f_i||_{L^p(\Omega)} \right] \]

Let us denote, for brevity,

\[ K_4 := (2S/\nu) \left( \sum_{i=1}^n ||d_i||_{L^p(\Omega_i)} + S||e^-||_{L^{np/(n+p)}(\Omega_i)} \right) \]

\[ K_5 := (2S/\nu) \left( \sum_{i=1}^n ||f_i||_{L^p(\Omega_i)} + S||f_0||_{L^{np/(n+p)}(\Omega_i)} \right) \]

and apply to (20) Hölder’s and Sobolev’s inequalities, thus obtaining

\[ ||u_t||_{L^1(\Omega)} \leq [\alpha(t)]^{(2+n)/2n} ||u_t||_{L^{2n/(n-2)}(\Omega_i)} \leq [\alpha(t)]^{1+ (p-n)/np} (K_4 t + K_5) \]

Now we follow a procedure of [1]. Put, for \( t \geq t_o \),

\[ \beta(t) := ||u_t||_{L^1(\Omega)} \]

and note that it turns out \( \beta(t) = \int_t^{+\infty} \alpha(s) \, ds \), therefore

\[ \beta'(t) = -\alpha(t) \leq 0 \quad \text{a.e. in } [t_o, +\infty) \]
From (22), (24) we deduce the differential inequality
\[
\beta(t) \leq (K_4 t + K_5)[-\beta'(t)]^{1+(p-n)/np} \quad \text{a.e. in } [t_0, +\infty)
\]
Suppose now, by contradiction, that \(\beta(t) > 0 \ \forall t \geq t_0\) (i.e., by definition of \(\beta(t)\), ess sup_{\Omega} u = +\infty\). Then in (25) we can divide by \(\beta(t)\) thus obtaining
\[
-\beta'(t)[\beta(t)]^{-np/(np+p-n)} \geq (K_4 t + K_5)^{-np/(np+p-n)}
\]
By integrating (26) from \(t_0\) and \(t^* > t_0\) (suppose for the moment \(K_4 > 0\), we obtain
\[
K_4[\beta(t_0)]^{(p-n)/(np+p-n)} - K_4[\beta(t^*)]^{(p-n)/(np+p-n)} \geq
\]
\[
\geq (K_4 t^* + K_5)^{(p-n)/(np+p-n)} - (K_4 t_0 + K_5)^{(p-n)/(np+p-n)}
\]
In (27) we get a contradiction when \(t^*\) tends to \(+\infty\), so it follows ess sup_{\Omega} u < +\infty. We can write again (27) with \(t_0 < t^* < \text{ess sup}_{\Omega} u\); by letting \(t^*\) tend to \(\text{ess sup}_{\Omega} u\) we deduce
\[
(K_4 \text{ess sup}_{\Omega} u + K_5)^{(p-n)/(np+p-n)} \leq
\]
\[
\leq (K_4 t_0 + K_5)^{(p-n)/(np+p-n)} + K_4[\beta(t_0)]^{(p-n)/(np+p-n)}
\]
Please note that the constant \(K_4\) is not greater than 2/3 because of (19). From (28) by easy calculations we get
\[
\text{ess sup}_{\Omega} u \leq (4/3)^{np/(p-n)}\|u_{t_0}\|_{L^1(\Omega_{t_0})} + 2^{np/(p-n)}t_0 + (3/2)[2^{np/(p-n)} - 1]K_5
\]
whence, by recalling the definition of \(t_0\) (given by (17)) and \(K_5\) (given by (21)) one can write
\[
\text{ess sup}_{\Omega} u \leq 2^{np/(p-n)}m + [(4/3)^{np/(p-n)} + 2^{np/(p-n)}\delta_0^{-1/2}]\|u_m\|_{L^2(\Omega)} +
\]
\[
+ (3S/\nu)[2^{np/(p-n)} - 1]\left[\sum_{i=1}^{n} \|f_i\|_{L^p(\Omega_{t_0})} + \sum_{i=1}^{n} \|f_i\|_{L^p(\Omega_{t_0})}\right]
\]
Finally, by taking into account (18), the definition of \(\delta_0\) (see (16)) and the functions \(\phi, \omega\), we conclude
\[
\text{ess sup}_{\Omega} u \leq 2^{np/(p-n)}m + [(4/3)^{np/(p-n)} + 2^{np/(p-n)}\delta_0^{-1/2}]\|u_m\|_{L^2(\Omega)} +
\]
\[
+ (3S/\nu)[2^{np/(p-n)} - 1]\left[\sum_{i=1}^{n} \phi(f_i, np/(p+n), \delta_0) + \sum_{i=1}^{n} \omega(f_i, np, \delta_0)\right]
\]
with \(\delta_0\) given by (16). □
Remark 4. If we suppose, besides the hypotheses of the preceding theorem, that there exists $q \geq 1$ such that $u_m \in L^q(\Omega)$, then we can write, instead of (15), (17)
\begin{equation}
\alpha(t) \leq ||u_m||_{L^q(\Omega_m)}^q (t - m)^{-q} \quad \forall t > m,
\end{equation}
\begin{equation}
t_o := m + ||u_m||_{L^q(\Omega)} \delta_o^{-1/q}
\end{equation}
and by proceeding as before we arrive at the conclusion in the form
\begin{equation}
\text{ess sup}_\Omega u \leq t_o + (np + (p - n)/(p - n)) K_5^{np/(np + p - n)} ||u_m||_{L^p(\Omega)} + (3S/\nu)^{2np/(np + p - n) - 1} [S \omega(f_o, np/(p + n), \delta_o) + \sum_{i=1}^n \omega(f, p, \delta_o)]
\end{equation}
where $\delta_o$ is always given by (16).

Remark 5. Suppose, in the bilinear form $a(\cdot, \cdot)$, that the coefficients $d_i$ and $c^{-}$ be identically zero. Then the constant $K_4$ defined by the first of (21) vanishes, and by integrating (25) we get, more simply,
\begin{equation}
\text{ess sup}_\Omega u \leq t_o + (np + p - n)/(p - n) K_5^{np/(np + p - n)} ||u_m||_{L^p(\Omega)}^{(p-n)/(np+p-n)}
\end{equation}
whence, by taking into account the definitions of $t_o$, $\delta_o$, ..., and Young's inequality, we deduce
\begin{equation}
\text{ess sup}_\Omega u \leq m + (\delta_o^{-1/2} + 1)||u_m||_{L^2(\Omega)} + [np/(p - n)] K_5
\end{equation}
This inequality is of the same kind of (31), but the coefficient of $m$ in it is now 1.

References


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