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Some Properties of the First Eigenvalue and the First Eigenfunction of Linear Second Order Elliptic Partial Differential Equations in Divergence Form.

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Sunto. – *Si studiano alcune proprietà del primo autovalore e delle autofunzioni delle equazioni differenziali alle derivate parziali ellittiche del secondo ordine, in forma variazionale e a coefficienti discontinui.*

1. – Introduction.

The present work is, in a certain sense, a continuation of [1]. By a small change in a lemma contained there, I prove some results concerning the first eigenvalue of linear second order elliptic partial differential equations in variational form with discontinuous coefficients. For example:

i) If A is an open set contained in $\Omega \subset R^n$ and different from Ω (in a proper sense), then it turns out

$$\lambda_1(A) < \lambda_1(\Omega)$$

where $\lambda_1(A)$, $\lambda_1(\Omega)$ denote the first eigenvalue of the equation concerning A, Ω respectively.

ii) With Ω fixed, the eigenfunction w_1 corresponding to the first eigenvalue λ_1 is unique (of course up to an arbitrary multiplicative constant).

For the sake of brevity I shall consider Dirichlet's problem only, but more general boundary problems could be studied as in [1].

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2. – Notations and hypotheses.

Let Ω be an open bounded connected set in R^n, where it is supposed $n \geq 3$ for simplicity.

Let $H^1(\Omega)$, $H_0^1(\Omega)$ be the (real) Hilbert spaces obtained by completing $C^1(\bar{\Omega})$, $C_0^1(\Omega)$ respectively according to the norm

$$\|u\|_{H^1(\Omega)} = \|u\|_{L_2(\Omega)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L_2(\Omega)} .$$

In $H_0^1(\Omega)$ an equivalent norm is the following:

$$\|u_x\|_{L_2(\Omega)} = \left\{\sum_{i=1}^{n} \|u_{x_i}\|^2_{L_2(\Omega)}\right\}^{\frac{1}{2}} .$$

For every function $u \in H_0^1(\Omega)$ the following inequality is satisfied

$$\|u\|_{L_{2n/(n-2)}(\Omega)} \leq K_1 \|u_x\|_{L_2(\Omega)} \tag{1}$$

where K_1 is a constant depending only on n (see e.g. [4], p. 487-488). If k is a real constant, $u \in H^1(\Omega)$ and B is a compact subset of $\bar{\Omega}$, we say that $u \leq k$ in B in the sense of $H^1(\Omega)$ if there exists a sequence

$\{u_j\}_{j\in N} \subset C^1(\bar{\Omega})$ such that: $u_j \leq k$ in B $(j = 1, 2, \ldots)$

and $\lim_j \|u - u_j\|_{H^1(\Omega)} = 0$.

Let B again be a compact subset of $\bar{\Omega}$.

We set

$$\operatorname{cap}_\Omega B = \inf\{\|v\|^2_{H^1(\Omega)} : v \in C^1(\bar{\Omega}),\ v \geq 1 \text{ in } B\} .$$

Let G be any subset of $\bar{\Omega}$; we define the internal capacity of G with respect to Ω in the following way:

$$\operatorname{cap}_{i,\Omega} G = \sup\{\operatorname{cap}_\Omega B : B \text{ compact},\ B \subset G\} .$$

(See in [2] some properties of the capacity with respect to Ω and a comparison with the ordinary capacity.)

Finally let us suppose: $a_{ij} \in L_\infty(\Omega)$, $\sum_{i,j=1}^{n} a_{ij} t_i t_j \geq \nu |t|^2$ a.e. in Ω, ν is a positive constant, $b_i \in L_n(\Omega)$, $d_i \in L_p(\Omega)$, $c \in L_{p/2}(\Omega)$

$(i, j = 1, 2, \ldots, n)$, $p > n$,

$$a(u, v) = \int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} (b_i u_{x_i} v + d_i u v_{x_i}) + cuv \right\} dx .$$

From the previous hypotheses and known results (see e.g. [7]) $a(\cdot, \cdot)$ is a bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$. Consider the Dirichlet problem

$$\text{(2)} \qquad \begin{cases} a(u, v) + \lambda(u, v)_{L_2(\Omega)} = 0 & \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega) . \end{cases}$$

Such a problem has only the trivial solution $u = 0$ in Ω when λ does not belong to a sequence $\{\lambda_j\}_{j\in N}$ of complex numbers which are called the eigenvalues of problem (2) (see [7], th. 3.4).

3. - Preliminary lemmata.

LEMMA 1. - *Suppose $w \in H^1(\Omega)$, $w \leqslant 0$ in Ω, w not identically zero in Ω, $a(w, v) \leqslant 0$ for any $v \in H_0^1(\Omega)$, $v \geqslant 0$ in Ω. Then*

$$\text{ess}_D \sup w < 0$$

for any compact subset D of Ω.

PROOF. - See [1], corollary 1. ∎

Let us consider now a slightly different version of Theorem 1 of [1].

LEMMA 2. - *The following two conditions are equivalent:*

(a) For any $u \in H^1(\Omega)$ such that $u \leqslant 0$ on $\partial\Omega$ in the sense of $H^1(\Omega)$ and $a(u, v) \leqslant 0$ for all $v \in H_0^1(\Omega)$, $v \geqslant 0$ in Ω, it turns out $u \leqslant 0$ in Ω.

(b) There exists (at least) one function $w \in H^1(\Omega)$ such that: $w \leqslant 0$ in Ω, $a(w, v) \leqslant 0$ for all $v \in H_0^1(\Omega)$, $v \geqslant 0$ in Ω, there exists a positive constant c_0 and a compact subset B of $\partial\Omega$ such that $w \leqslant -c_0$ in B in the sense of $H^1(\Omega)$ and $\text{cap}_\Omega B > 0$.

PROOF. - It is nearly the same of [1] and I repeat it for readers' convenience only.

(*a*) ⇒ (*b*). Suppose $f \in C^1(\bar{\Omega})$, $f < 0$ on $\partial\Omega$.
Consider the following Dirichlet problem:

$$(3) \qquad \begin{cases} a(w, v) = 0 , \\ w - f \in H_0^1(\Omega) . \end{cases}$$

From hypothesis (*a*) such a problem has at most one solution w. But the Riesz-Fredholm theory is valid for it (see e.g. [7]), so the uniqueness of the solution w implies its existence. It is easy to verify that w satisfies condition (*b*) (see Lemma 4).

(*b*) ⇒ (*a*). Let $u \in H^1(\Omega)$, $u \leqslant 0$ on $\partial\Omega$ in the sense of $H^1(\Omega)$, $a(u, v) \leqslant 0$ for any $v \in H_0^1(\Omega)$, $v \geqslant 0$ in Ω; let us show that in this case $u \leqslant 0$ in Ω.
Let us consider the function

$$w_k = \max(u + kw, 0)$$

where k is a real number and w is the function satisfying hypothesis (*b*). Put

$$\Omega(k) = \{x \colon x \in \Omega,\ w_k(x) > 0\} ,$$

$$k_0 = \inf\{k \colon w_k = 0 \text{ in } \Omega\} .$$

The result will be proved if we show that $k_0 \leqslant 0$: suppose $k_0 > 0$ in order to find a contradiction. Consider a compact subset D contained in Ω; from Lemma 1 it follows

$$\operatorname{ess}_D \sup w < 0$$

and from [7] (par. 5)

$$\operatorname{ess}_D \sup u < +\infty .$$

It follows that there exists a number h such that

$$(4) \qquad w_k = 0 \text{ in } D , \qquad \forall k \geqslant h .$$

Since D is arbitrary, we get

$$(5) \qquad \lim_{k \to +\infty} \operatorname{mis} \Omega(k) = 0 .$$

Let us show now that, even if k_0 is finite, it turns out

$$(6) \qquad \lim_{k \to k_0^-} \operatorname{mis} \Omega(k) = 0 .$$

In fact if (6) is not verified there would exist a subset E contained in Ω such that $\operatorname{mis} E > 0$, $u + k_0 w = 0$ in E, $u + k_0 w \leqslant 0$ in Ω. From Lemma 1 applied to the function $u + k_0 w$ it would follow $u + k_0 w = 0$ in Ω. That is $-u = k_0 w$, whence $w \geqslant 0$ on $\partial\Omega$ in the sense of $H^1(\Omega)$, a contradiction because it is $w \leqslant -c_0$ in B and $\operatorname{cap}_\Omega B > 0$ (hypothesis (b)). So (6) is true. Now suppose $0 < k < k_0$, so that $w_k \geqslant 0$ in Ω, $w_k = 0$ in $\Omega - \Omega(k)$, whence

$$a(u + kw, w_k) = a(w_k, w_k) \leqslant 0 . \tag{7}$$

From the assumptions on the coefficients and Hölder's inequality we get

$$\begin{aligned}(8)\qquad \nu \|(w_k)_x\|^2_{L_2(\Omega(k))} \leqslant \sum_{i=1}^{n} \|b_i + d_i\|_{L_n(\Omega(k))} \cdot \|w_k\|_{L_{2n/(n-2)}(\Omega(k))} \cdot \\ \cdot \|(w_k)_x\|_{L_2(\Omega(k))} + \|c\|_{L_{n/2}(\Omega(k))} \cdot \|(w_k)_x\|^2_{L_{2n/(n-2)}(\Omega(k))} .\end{aligned}$$

From (1), (8) it follows

$$\begin{aligned}(9)\qquad \nu \|(w_k)_x\|^2_{L_2(\Omega(k))} \leqslant \\ \leqslant K_1\Big[\sum_{i=1}^{n} \|b_i + d_i\|_{L_n(\Omega(k))} + \|c\|_{L_{n/2}(\Omega(k))}\Big] \cdot \|(w_k)_x\|^2_{L_2(\Omega(k))} .\end{aligned}$$

From (6) and (9) we deduce that if $k_0 - k$ is sufficiently small (with $k < k_0$) it turns out $(w_k)_x = 0$ in Ω, that is $w_k = 0$ in Ω, a contradiction. Therefore $k_0 = 0$. ∎

LEMMA 3. – *Let A be an open set contained in Ω. Then $H^1_0(A) = H^1_0(\Omega)$ if and only if* $\operatorname{cap}_{i,A}(\partial A - \partial\Omega) = 0$.

REMARK. – The equality $H^1_0(A) = H^1_0(\Omega)$ must be understood in the sense that the following two conditions are satisfied:

(i) For any $\varphi \in H^1_0(A)$ the function

$$\tilde\varphi(x) = \begin{cases} \varphi(x) , & x \in A , \\ 0 , & x \in \Omega - A \end{cases}$$

belongs to $H^1_0(\Omega)$. This is obviously true since $A \subset \Omega$.

(ii) For any $\psi \in H^1_0(\Omega)$ it turns out $\psi = 0$ on ∂A in the sense of $H^1_0(A)$.

PROOF. – Let us suppose $\operatorname{cap}_{i,A}(\partial A - \partial\Omega) > 0$. From the definition of $\operatorname{cap}_{i,A}(\partial A - \partial\Omega)$ there exists a compact set D such that:

$D \subset \partial A - \partial\Omega$ (whence $D \subset \Omega$), $\mathrm{cap}_A D > 0$. Let z be a function such that $z \in C_0^1(\Omega)$, $z > 0$ in D. Obviously it is $z > 0$ in D in the sense of $H^1(A)$ also, therefore $z \notin H_0^1(A)$. This proves that in this case $H_0^1(A) \neq H_0^1(\Omega)$.

Conversely suppose $\mathrm{cap}_{i,A}(\partial A - \partial\Omega) = 0$ and $v \in H_0^1(\Omega)$: in this case we shall show that $v \in H_0^1(A)$.

In fact let $\{v_j\}_{j\in N}$ be a sequence of functions such that:

$$v_j \in C_0^1(\Omega)\,, \qquad \lim_j \|v - v_j\|_{H^1(\Omega)} = 0\,.$$

For any $\varepsilon > 0$, set

$$\Omega(\varepsilon, j) = \{x\colon x \in \bar{\Omega},\ v_j(x) \geqslant \varepsilon\}\,, \qquad (j = 1, 2, \ldots).$$

Obviously $\Omega(\varepsilon, j) \subset \Omega$ so that $\Omega(\varepsilon, j) \cap \partial A$ are compact subsets of Ω. From the assumption $\mathrm{cap}_{i,A}(\partial A - \partial\Omega) = 0$ it follows

$$\mathrm{cap}_A\,(\Omega(\varepsilon, j) \cap \partial A) = 0\,, \qquad (j = 1, 2, \ldots).$$

Therefore there exist functions $g_j \in C^1(\bar{A})$ such that:

$$\|g_j\|_{H^1(A)} < \frac{1}{j}\,, \qquad g_j \leqslant 0 \text{ in } \bar{A}\,, \qquad g_j \leqslant -v_j \text{ in } \Omega(\varepsilon, j) \cap \partial A \qquad (j = 1, 2, \ldots).$$

It turns out:

$$\lim_{j\to+\infty} \|v - v_j - g_j\|_{H^1(A)} = 0\,, \qquad v_j + g_j \leqslant 0 \text{ in } \partial A \cap \Omega(\varepsilon, j),$$

$v_j + g_j \leqslant \varepsilon$ in $\partial A - \Omega(\varepsilon, j)$. That is $v_j + g_j \leqslant \varepsilon$ on ∂A $(j = 1, 2, \ldots)$, which implies $v \leqslant \varepsilon$ on ∂A in the sense of $H^1(A)$. Since ε is arbitrary, it follows $v \leqslant 0$ on ∂A in the sense of $H^1(A)$. Similarly it can be proven that $v \geqslant 0$ on ∂A in the sense of $H^1(A)$; therefore $v \in H_0^1(A)$. ■

LEMMA 4. – *For every open set Ω in R^n it turns out*

$$\mathrm{cap}_\Omega\, \partial\Omega > 0\,.$$

PROOF. – The result can be deduced from the following inequality:

$$\text{(10)} \qquad \mathrm{mis}\,\Omega \leqslant \left[\frac{\|v\|_{H^1(\Omega)}}{k}\right]^2 + \left[\frac{K_1\|v\|_{H^1(\Omega)}}{1-k}\right]^{2n/(n-2)}$$

valid for every real number k such that $0<k<1$ and for every function $v\in C^1(\bar{\Omega})$ such that $0\leqslant v\leqslant 1$ in $\bar{\Omega}$, $v=1$ on $\partial\Omega$.

Let us prove (10). Choose k and v as before and set

$$\Omega(k)=\{x\colon x\in\bar{\Omega},\ v(x)\geqslant k\}\,,\qquad 2^*=2n/(n-2)\,.$$

It turns out:

$$\|v\|^2_{H^1(\Omega)}\geqslant\int_\Omega v^2\,dx\geqslant\int_{\Omega(k)}v^2\,dx\geqslant k^2\operatorname{mis}\Omega(k)\,. \tag{11}$$

Besides obviously $1-v\in C^1(\bar{\Omega})\cap H^1_0(\Omega)$; so we can apply (1) to the function $1-v$. We find

$$\|1-v\|_{L_{2^*}(\Omega)}\leqslant K_1\|(1-v)_x\|_{L_2(\Omega)}=K_1\|v_x\|_{L_2(\Omega)}\leqslant K_1\|v\|_{H^1(\Omega)}\,. \tag{12}$$

From (12) it follows

$$(1-k)^{2^*}\operatorname{mis}[\Omega-\Omega(k)]\leqslant\int_{\Omega-\Omega(k)}(1-v)^{2^*}dx\leqslant[K_1\|v\|_{H^1(\Omega)}]^{2^*}\,. \tag{13}$$

From (11), (13) finally

$$\operatorname{mis}\Omega=\operatorname{mis}\Omega(k)+\operatorname{mis}[\Omega-\Omega(k)]\leqslant\left[\frac{\|v\|_{H^1(\Omega)}}{k}\right]^2+\left[\frac{K_1\|v\|_{H^1(\Omega)}}{1-k}\right]^{2^*},$$

that is (10). From it and from the definition of capacity with respect to Ω the thesis of the present lemma can easily be deduced. ∎

4. - The first eigenvalue.

Let us denote by $\lambda_1(\Omega)$ the eigenvalue of problem (2) having maximum real part.

From the results of [1] $\lambda_1(\Omega)$ is real. Suppose A is an open set contained in Ω and consider the analogous of problem (2) in A:

$$\begin{cases} a(z,v)+(z,v)_{L_2(A)}=0 & \forall v\in H^1_0(A),\\ z\in H^1_0(A)\,. \end{cases} \tag{14}$$

Denote by $\lambda_1(A)$ the eigenvalue of problem (14) having maximum real part.

THEOREM 1. - *It turns out* $\lambda_1(A)\leqslant\lambda_1(\Omega)$.

A necessary and sufficient condition in order that $\lambda_1(\Omega)=\lambda_1(A)$ *is*

$$\operatorname{cap}_{i,A}(\partial A-\partial\Omega)=0\,.$$

PROOF. – The first inequality can be deduced in the following way.

From [1] (Theorem 2) we get

$$\lambda_1(\Omega) = -\sup\left\{\inf_{\substack{v\in H_0^1(\Omega)\\ v>0}} \frac{a(w, v)}{(w, v)_{L_2(\Omega)}} : w\in H^1(\Omega),\ w>0 \text{ in } \Omega\right\}.$$

If follows that if $\lambda > \lambda_1(\Omega)$ there exists (at least) a function $z\in H^1(\Omega)$ such that:

$$z<0 \text{ in } \Omega,\ a(z, v)+\lambda(z, v)_{L_2(\Omega)}<0 \text{ for any } v\in H_0^1(\Omega),\ v>0 \text{ in } \Omega.$$

Since $A\subset\Omega$, it turns out obviously $a(z, v)+\lambda(z, v)_{L_2(A)}<0$ for any $v\in H_0^1(A)$, $v>0$ in A. So from [1] (Theorem 1 and Corollary 2) neither λ nor the numbers greater than λ are eigenvalues of problem (1).

It follows $\lambda_1(A)<\lambda$ and therefore $\lambda_1(A)\leqslant\lambda_1(\Omega)$.

Let us prove now that if $\text{cap}_{i,A}(\partial A-\partial\Omega)>0$ then $\lambda_1(A)<\lambda_1(\Omega)$. Denote by $w_1\in H_0^1(\Omega)$ an eigenfunction corresponding to the eigenvalue $\lambda_1(\Omega)$: it turns out

$$(15)\qquad a(w_1, v)+\lambda_1(\Omega)(w_1, v)_{L_2(\Omega)}=0\,, \qquad \forall v\in H_0^1(\Omega).$$

From [5] (Theorem 6.1) w_1 can be chosen such that $w_1\leqslant 0$ in Ω and therefore (Lemma 1) $w_1<0$ in every compact subset D of Ω. Since by hypothesis $\text{cap}_{i,A}(\partial A-\partial\Omega)>0$, there exists a compact subset B of $\partial A-\partial\Omega$ (and of Ω) such that $\text{cap}_A B>0$.

We can apply Lemma 2 to the open set A.

In fact from (15) we get

$$(16)\qquad a(w_1, v)+\lambda_1(\Omega)(w_1, v)_{L_2(A)}=0\,, \qquad \forall v\in H_0^1(A),$$

and $w_1<0$ in B with $\text{cap}_A B>0$. Since w_1 is continuous in Ω (see [7]), we can say more precisely that $w_1\leqslant -c_0$ in B in the sense of $H^1(\Omega)$, with c_0 a positive constant. From Lemma 2 (in the sense $(b)\Rightarrow(a)$) it follows that neither $\lambda_1(\Omega)$ nor the numbers greater than $\lambda_1(\Omega)$ are eigenvalues of problem (15).

It remains to show that if $\text{cap}_{i,A}(\partial A-\partial\Omega)=0$ then $\lambda_1(\Omega)=\lambda_1(A)$. To this purpose it is sufficient to observe that in this case from Lemma 3 it is $H_0^1(A)=H_0^1(\Omega)$, so that $w_1\in H_0^1(A)$ and w_1 is an eigenfunction. From (16) it follows that $\lambda_1(\Omega)$ is an eigenvalue for problem (14), whence $\lambda_1(\Omega)\geqslant\lambda_1(A)$ and, from what we proved previously, $\lambda_1(\Omega)=\lambda_1(A)$. ∎

5. – Uniqueness of the first eigenfunction.

THEOREM 2. – *The linear space of the eigenfunctions corresponding to the first eigenvalue* $\lambda_1(\Omega)$ *has dimension* 1.

PROOF. – For the sake of contradiction let us suppose that there exist two linearly independent eigenfunctions $w_1, \hat{w}_1 \in H_0^1(\Omega)$:

$$\begin{cases} a(w_1, v) + \lambda_1(\Omega)(w_1, v)_{L_2(\Omega)} = 0\,, \\ a(\hat{w}_1, v) + \lambda_1(\Omega)(\hat{w}_1, v)_{L_2(\Omega)} = 0 \end{cases} \qquad \forall v \in H_0^1(\Omega). \tag{17}$$

Then it is possible to find a linear combination $z = c_1 w_1 + c_2 \hat{w}_1$ such that $\{x: x \in \Omega,\ z(x) > 0\}$ and $\{x: x \in \Omega,\ z(x) < 0\}$ are non void open sets (in fact z is a continuous function in Ω). Let S be an open ball such that $\bar{S} \in \{x: x \in \Omega,\ z(x) < 0\}$.

Put $A = \Omega - \bar{S}$; it is easy to verify that $\operatorname{cap}_A \bar{S} > 0$.

Besides, from the already used Theorem 6.1 of [5], there exists an eigenfunction negative in Ω: we can suppose that such an eigenfunction coincides with w_1. So it is possible to apply Lemma 2 to the open set A in the sense $(b) \Rightarrow (a)$: in fact it is $w_1 < 0$ in $\bar{S}$, $\operatorname{cap}_A \bar{S} > 0$, $z \leqslant 0$ on ∂A in the sense of $H^1(A)$, $a(z, v) + \lambda(\Omega)(z, v)_{L_2(A)} = 0$ $\forall v \in H_0^1(A)$.

From Lemma 2 we conclude that $z \leqslant 0$ in A, a contradiction since we supposed $\{x: x \in \Omega,\ z(x) > 0\} \neq \emptyset$. ∎

About the other eigenfunctions, a simple remark is the following.

PROPOSITION. – *Let* w *be any eigenfunction in* Ω *corresponding to a real eigenvalue* λ.

Then the set $\{x: x \in \Omega,\ w(x) \neq 0\}$ *has a finite number of connected components.*

PROOF. – From the hypotheses we get

$$\begin{cases} a(w, v) + \lambda(w, v)_{L_2(\Omega)} = 0 & \forall v \in H_0^1(\Omega), \\ w \in H_0^1(\Omega)\,. \end{cases} \tag{18}$$

Let A be a nonempty connected component of $\{x: x \in \Omega,\ w(x) \neq 0\}$. Remembering the continuity of w in Ω, it turns out $w = 0$ on ∂A in the sense of $H^1(A)$.

If we set

$$w_A = \begin{cases} w & \text{in } A \\ 0 & \text{in } R^n - A \end{cases}$$

it follows $w_A \in H_0^1(A)$.

So we can put $v = w_A$ in (18) obtaining

$$a(w_A, w_A) + \lambda(w_A, w_A)_{L_2(A)} = 0 . \tag{19}$$

From (19), (1) and Hölder's inequality we find

$$\nu \|(w_A)_x\|^2_{L_2(A)} \leq K_1 \Big[\sum_{i=1}^{n} \|b_i + d_i\|_{L_n(A)} + K_1 \|c\|_{L_{n/2}(A)} + |\lambda| K_1 (\text{mis}\, A)^{2/n} \Big] \|(w_A)_x\|^2_{L_2(A)} . \tag{20}$$

From (20) it follows that the measure of A cannot be too small, because otherwise it would be $w_x = 0$ in A, a contradiction.

This is sufficient to conclude that the open sets like A are in a finite number. ▪

In the autoadjoint case (i.e. if $a_{ij} = a_{ji}$, $b_i = d_i$, for $i, j = 1, 2, \ldots, n$) further results can be found in [6].

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