An a priori inequality concerning elliptic second order partial differential equations of variational type

MAUROZIO CRICCO (*) (**) 

SUMMARY. — I give an explicit evaluation of the constant appearing in an a priori inequality for the solutions of linear second order elliptic partial differential equations in divergence form. An application is also considered.

1. Introduction.

About the theory of linear second order elliptic partial differential equations in divergence form with discontinuous coefficients, studied by G. Stampacchia in [3], the following maximum principle is known (see [1], [3]):

Suppose \( u \in H^2(\Omega) \),

\[
\sigma(\alpha, \beta) = \sum_{i,j=1}^{n} \sum_{k=1}^{n} a_{ij} \alpha_i \alpha_j + \sum_{j=1}^{n} (b_k \alpha_i \beta_j + d_k \alpha_i \alpha_j) + \alpha \varphi \leq 0
\]

for any \( \beta \in H_0^1(\Omega), \beta \geq 0 \) in \( \Omega \). Under suitable hypotheses on the coefficients \( a_{ij}, b_k, d_k \), \( \sigma(\alpha, \beta) \leq 0 \) if it follows

\[
\sigma \sup_{\Omega} \beta \leq \max_{\Omega} \varphi.
\]

(*) Entra in Redazione il 04-04-1971.
(**) Lavoro eseguito nell'ambito dei « Centro di Matematica e Fisica Teorica » del C.N.R. presso l'Università di Genova.
Given any \( f_1, f_2, \ldots, f_n \in L^2(\Omega) \), this result implies uniqueness and existence of the solution \( w \) of the Dirichlet problem

\[
\begin{cases}
    a(w, v) = \sum_{i=1}^{n} f_i v_i \, dx \\
    w \in H^1_0(\Omega).
\end{cases}
\]

(1)

From the closed graph theorem it follows also the existence of a constant \( K \) such that the solution \( w \) of (1) satisfies the inequality

\[
\| w \|_{H^1(\Omega)} \leq K \sum_{i=1}^{n} \| f_i \|_{L^2(\Omega)}.
\]

Nevertheless, as far as I know, the problem of finding the dependence of this constant \( K \) on the coefficients is open (except the trivial case when the bilinear form \( a(\cdot, \cdot) \) is coercive on \( H^1_0(\Omega) \), i.e. \( a(\cdot, \cdot) \geq k \| \cdot \|_{H^1(\Omega)}^2 \) \( \forall x \in H^1_0(\Omega) \) with \( k \) positive constant).

The aim of the present work is to give an upper bound of \( K \) through the ellipticity constant \( \nu \), the norms in \( L^2_w(\Omega) \) of the functions \( b_i - d_i \) (\( i = 1, 2, \ldots, n \)) and the number of dimensions \( n \). Although the value I find for \( K \) is not the best possible, the result is sufficient for some applications.

2. Notations and hypotheses.

Let \( \Omega \) be an open bounded set of \( \mathbb{R}^n \); we suppose for simplicity \( n \geq 2 \).

Let us denote by \( H^1(\Omega) \) the space obtained by completing \( C_c^1(\Omega) \) according to the norm

\[
\| u \|_{H^1(\Omega)} = \| u \|_{L^2(\Omega)} + \sum_{i=1}^{n} \| w_i \|_{L^2(\Omega)}.
\]

Let \( H^1_0(\Omega) \) be the following subspace of \( H^1(\Omega) : 

\[
H^1_0(\Omega) = \text{closure of } C_c^1(\Omega) \text{ in } H^1(\Omega).
\]

In \( H^1_0(\Omega) \) we can assume as a norm the expression

\[
\| u \|_{L^2(\Omega)} = \left( \sum_{i=1}^{n} \| w_i \|_{L^2(\Omega)}^2 \right)^{1/2}.
\]

In this space the norms (3) and (3) are equivalent, as the following lemma claims.

**Lemma 1.** For any \( u \in H^1_0(\Omega) \) it results

\[
\| u \|_{L^2(\Omega)} \leq \| u \|_{H^1(\Omega)},
\]

where

\[
S = \frac{2}{n(n-1)} \sum_{i=1}^{n} \left( \frac{n}{2} \Gamma \left( \frac{n}{2} \right) \right)^{1/n}.
\]

**Proof:** see e.g. [2] page 688. We observe that the constant \( S \) depends only on \( n \) and is the best possible.

Then we suppose

\[
\alpha_i \in L^\infty(\Omega), \quad \sum_{i=1}^{n} \alpha_i \leq \beta \quad \text{a.e. in } \Omega,
\]

\( \nu \) is a positive constant,

\[
b_i, \quad d_i \in L^\infty(\Omega), \quad (i, j = 1, 2, \ldots, n),
\]

\( \sigma \in L^\infty(\Omega), \quad \sigma = \sum_{i=1}^{n} (d_i w_i) \geq 0 \) in the sense of distributions,

\[
\alpha(u, v) = \sum_{i,j=1}^{n} \alpha_{ij} u_i v_j + \sum_{i=1}^{n} (b_i u_i v + d_i u_i w_i + cu) \, dx.
\]

3. Main result.

**Theorem 1.** We suppose that the hypotheses described above are satisfied, and moreover \( f_i \in L^2(\Omega) \) (\( i = 1, 2, \ldots, n \)), \( w \in H^1_0(\Omega) \).

Then

\[
\| w \|_{L^2(\Omega)} \leq \frac{2S \Gamma \left( \frac{n}{2} \right)}{\nu} \left( \sum_{i=1}^{n} \| f_i \|_{L^2(\Omega)}^2 \right)^{1/2}.
\]
where \( p \) is the smallest integer greater than
\[
\left( \frac{2p}{p-1} \right) \sum_{i=1}^{n} ||b_i - d_i||_{L^2(\Omega)}^p.
\]

**Proof.** The result is trivial if
\[
\sum_{i=1}^{n} ||b_i - d_i||_{L^2(\Omega)} \leq \frac{\gamma}{2} \delta
\]
(where \( \delta \) is the constant defined in lemma 1).

In fact let us proceed as in [3]: putting \( v = w \) in (4) we get
\[
\sum_{j=1}^{n} \int_{\Omega} f_j \cdot w_j \, dx = a(w, w) = \int_{\Omega} \left( \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} w_j w_k + \sum_{i=1}^{n} (b_i - d_i) w_i w_i \right) \, dx = \int_{\Omega} \left( \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} w_j w_k + \sum_{k=1}^{n} (b_k - d_k) w_k w_k + \left( s - \sum_{k=1}^{n} (d_k)^2 \right) w_k^2 \right) \, dx.
\]

Now remembering lemma 1, the hypotheses listed above and Hölder's inequality we get from (6):
\[
\sum_{i=1}^{n} \int_{\Omega} f_i \cdot w_i \, dx \leq \gamma \sum_{i=1}^{n} ||w_i||_{L^2(\Omega)}^2 \sum_{i=1}^{n} ||b_i - d_i||_{L^2(\Omega)} ||w_i||_{L^2(\Omega)} \leq \sum_{i=1}^{n} \int_{\Omega} f_i \cdot w_i \, dx
\]
whence at ones
\[
||w||_{L^2(\Omega)} \leq 2\gamma \left( \sum_{i=1}^{n} ||f_i||_{L^2(\Omega)}^2 \right)^{1/2}.
\]

Now let us suppose \( \sum_{i=1}^{n} ||b_i - d_i||_{L^2(\Omega)} > \gamma/2 \delta \). Set
\[
\alpha_i = \max(\epsilon - k_i, 0) + \min(\epsilon + k_i, 0).
\]
\[
\Omega_i = \left\{ x \in \Omega : \frac{\partial \alpha_i}{\partial x_i}(x) > 0 \right\}
\]

where \( k_i \) is any positive number. Since this function \( \omega_i \) belongs to \( L^2(\Omega_i) \) we can put \( v = \omega_i \) in (4) and we get
\[
(3) \quad a(\omega_i, \omega_i) = \int_{\Omega_i} \left( \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} \omega_j \omega_k + \sum_{i=1}^{n} (b_i - d_i) \omega_i \omega_i + \sum_{k=1}^{n} (d_k)^2 \omega_k^2 \right) \, dx.
\]

We observe now that \( \omega_i \omega_i \geq 0 \) in \( \Omega \), and in the set where \( \omega_i = 0 \) it is
\[
\omega_i = \lambda \omega_i \quad (i = 1, 2, \ldots, n).
\]

So from (3) it follows
\[
(9) \quad a(\omega_i, \omega_i) \geq \int_{\Omega_i} \left( \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} \omega_j \omega_k + \sum_{i=1}^{n} (b_i - d_i) \omega_i \omega_i \right) \, dx.
\]

From (4), (9) we have
\[
(10) \quad \gamma ||\omega_i||_{L^2(\Omega_i)}^2 \leq \gamma \sum_{i=1}^{n} ||b_i - d_i||_{L^2(\Omega_i)} ||\omega_i||_{L^2(\Omega_i)} + \sum_{k=1}^{n} \left( \frac{1}{2} \delta \right) ||\omega_i||_{L^2(\Omega_i)}.
\]

Now it is easy to see that the measure of \( \Omega_i \) is a continuous function of \( k_i \), so that \( \sum_{i=1}^{n} ||b_i - d_i||_{L^2(\Omega_i)} \) also is a continuous function of \( k_i \). Furthermore it is
\[
(11) \quad \lim_{k_i \to +\infty} \text{mes} \, \Omega_i = 0.
\]

From all these facts it follows that \( k_i \) can be fixed in such a way that
\[
(12) \quad \delta \sum_{i=1}^{n} ||b_i - d_i||_{L^2(\Omega_i)} = \gamma/2.
\]

From (10), (12) we get
\[
(13) \quad ||\omega_i||_{L^2(\Omega_i)} \leq 2/\gamma \left( \sum_{i=1}^{n} ||f_i||_{L^2(\Omega_i)}^2 \right)^{1/2}.
\]
Now suppose $0 \leq k_2 < k_1$ and consider the function

$$w_2 = \min \left[ k_1 - k_2, \max (w - k_2, 0) \right] + \max [k_2 - k_1, \min (w + k_2, 0)]$$

and

$$\Omega_2 = \left\{ x : x \in \Omega, \sum_{i=1}^{n} \frac{\partial w_2}{\partial x_i} (x) > 0 \right\}.$$

Since $w_2$ also belongs to $N^1_a (\Omega)$ putting $v = w_2$ in (4) we get

$$\sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial f_i (w_2)_{x_j}}{\partial x_i} \, dx = \int_{\Omega} \left[ \sum_{i,j=1}^{n} \partial_i w_2 \partial_j (w_2)_{x_j} + \sum_{i=1}^{n} (k_1 - d_i) w_2 \partial_i w_2 \right] + \int_{\Omega} \left[ a - \sum_{i=1}^{n} (k_1 - d_i) \right] w_2 \, dx.$$

Observe that $w_2 \geq 0$ in $\Omega$, $w_{x_1} w_2 = (w_1 + w_2)_{x_1} w_2$ in $\Omega$,

$$w_{x_i} = (w_2)_{x_i} \quad (i = 1, 2, \ldots, n).$$

Therefore from (14) it follows

$$\int_{\Omega} \left[ \sum_{i,j=1}^{n} \partial_i w_2 \partial_j (w_2)_{x_j} + \sum_{i=1}^{n} (k_1 - d_i) w_2 \partial_i w_2 \right] \, dx + \int_{\Omega} \left[ a - \sum_{i=1}^{n} (k_1 - d_i) \right] w_2 \, dx \leq \sum_{i=1}^{n} \int_{\Omega} f_i (w_2)_{x_i} \, dx$$

whence, from Hölder’s inequality and lemma 1:

$$\left( \sum_{i=1}^{n} \int_{\Omega} \left[ f_i (w_2)_{x_i} \right]^2 \, dx \right)^{1/2} \leq \left( \sum_{i=1}^{n} \int_{\Omega} \left[ f_i (w_2)_{x_i} \right]^2 \, dx \right)^{1/2} \leq \sum_{i=1}^{n} \int_{\Omega} \left[ f_i (w_2)_{x_i} \right]^2 \, dx$$

The measure of $\Omega_2$ is a continuous function of $k_2$, and it is easy to see that $\lim_{k_2 \to k_1} \mu_2 = 0$. Therefore it is possible to choose $k_2$ so that

$$\sum_{i=1}^{n} \int_{\Omega} \left[ b_i - d_i \right] \, dx = \gamma / 2$$

or $k_2 = 0$ if, with this value, it results

$$\sum_{i=1}^{n} \int_{\Omega} \left[ b_i - d_i \right] \, dx = \gamma / 2.$$

From (12), (15), (16) it follows

$$\left( \sum_{i=1}^{n} \int_{\Omega} \left[ f_i (w_2)_{x_i} \right]^2 \, dx \right)^{1/2} \leq \gamma / 2 \left( \sum_{i=1}^{n} \int_{\Omega} \left[ f_i (w_2)_{x_i} \right]^2 \, dx \right)^{1/2}.$$

Continuing the same procedure, let us consider in general some numbers

$$0 = k_0 < k_{p-1} < \ldots < k_q < k_{q-1} < \ldots < k_1.$$

We define

$$w_q = \min \left[ k_{q-1} - k_q, \max (w - k_q, 0) \right] + \max \left[ k_q - k_{q-1}, \min (w + k_q, 0) \right],$$

$$\Omega_q = \left\{ x : x \in \Omega, \sum_{i=1}^{n} \frac{\partial w_q}{\partial x_i} (x) > 0 \right\},$$

and fix every number $k_q$ such that

$$\sum_{i=1}^{n} \int_{\Omega} \left[ b_i - d_i \right] \, dx = \gamma / 2 \left( \sum_{i=1}^{n} \int_{\Omega} \left[ f_i (w_q)_{x_i} \right]^2 \, dx \right)^{1/2}$$

and

$$\sum_{i=1}^{n} \int_{\Omega} \left[ b_i - d_i \right] \, dx = \gamma / 2 \left( \sum_{i=1}^{n} \int_{\Omega} \left[ f_i (w_q)_{x_i} \right]^2 \, dx \right)^{1/2}.$$

The integer $p$ is therefore determined as the first index for which (19) is valid with $k_q = 0$.

Substituting $v = w_q$ in (4), with easy computations we find

$$\int_{\Omega} \left[ \sum_{i,j=1}^{n} \partial_i w_q \partial_j (w_q)_{x_j} + \sum_{i=1}^{n} (k_1 - d_i) w_q \partial_i w_q \right] \, dx = \int_{\Omega} \left[ \sum_{i,j=1}^{n} \partial_i w_q \partial_j (w_q)_{x_j} + \sum_{i,j=1}^{n} \partial_i w_q \partial_j (w_q)_{x_j} \right] \, dx$$

and

$$\left( \sum_{i=1}^{n} \int_{\Omega} \left[ f_i (w_q)_{x_i} \right]^2 \, dx \right)^{1/2} \leq \gamma / 2 \left( \sum_{i=1}^{n} \int_{\Omega} \left[ f_i (w_q)_{x_i} \right]^2 \, dx \right)^{1/2}.$$
From Schwartz-Hölder's inequality and lemma 1 it follows

\[ r \| (w_{q}h) \|_{L^{q}(\Omega)} \leq \frac{n}{r} \sum_{i=1}^{n} \| b_i - d_i \|_{L^{q}(\Omega)} \| (w_{q}h) \|_{L^{r}(\Omega)} + \]

\[ \left( \sum_{i=1}^{n} \| f_i \|_{L^{r}(\Omega)} \right)^{\frac{r}{r'}} + \left( \sum_{i=1}^{n} \| b_i \|_{L^{q}(\Omega)} \| (w_{q}h) \|_{L^{r}(\Omega)} \right)^{\frac{q}{q'}}. \]

and from (18), (19) finally

\[ (20) \quad \| (w_{q}h) \|_{L^{q}(\Omega)} \leq \frac{n}{r} \left( \sum_{i=1}^{n} \| f_i \|_{L^{r}(\Omega)} \right)^{\frac{r}{r'}} + \left( \sum_{i=1}^{n} \| b_i \|_{L^{q}(\Omega)} \| (w_{q}h) \|_{L^{r}(\Omega)} \right)^{\frac{q}{q'}}. \]

\[ (q = 1, 2, \ldots, p). \]

This inequality permits to prove what we want.

In fact it is easy to deduce from (20) that

\[ \| w_{q} \|_{L^{q}(\Omega)} \leq \frac{n}{r} \left( \sum_{i=1}^{n} \| f_i \|_{L^{r}(\Omega)} \right)^{\frac{r}{r'}}. \]

whence, observing that \( w_{q} = \sum_{q=1}^{p} (w_{q}h)_{q} \):

\[ (21) \quad \| w_{q} \|_{L^{q}(\Omega)} \leq \frac{n}{r} \left( \sum_{i=1}^{n} \| f_i \|_{L^{r}(\Omega)} \right)^{\frac{r}{r'}}. \]

With (21) the result is proven if we evaluate \( p \).

From the hypotheses and easy inequalities we get

\[ (p - 1) \frac{r}{2p} = \frac{n}{r} \sum_{i=1}^{n} \| b_i - d_i \|_{L^{q}(\Omega)} \leq (p - 1)^{\frac{r}{r'}} \sum_{i=1}^{n} \| b_i - d_i \|_{L^{q}(\Omega)}. \]

whence at once

\[ p \leq 1 + \left( \frac{2p}{r} \sum_{i=1}^{n} \| b_i - d_i \|_{L^{q}(\Omega)} \right)^{\frac{r}{r'q}}. \]

4. Example of applications.

Let us use the preceding result to prove the existence of the solution of a Dirichlet problem in an unbounded domain. Consider an open unbounded set \( D \) in \( \mathbb{R}^n \), with \( \mathbb{R}^n - D \neq \emptyset \); let \( V \) denote the space obtained by completing \( G^1_0(D) \) according to the norm

\[ \| u \|_{L^{2}(\Omega)} = \left( \sum_{i=1}^{n} \| u_i \|_{L^{2}(\Omega)} \right)^{\frac{1}{2}}. \]

\( V \) is an Hilbert space with the scalar product

\[ (u, v)_{\Omega} = \sum_{i=1}^{n} \int_{\Omega} u_i \partial_{i} v \, dx. \]

From lemma 1 we get \( V \subset L^{(p-1)/2}(\Omega) \), whence \( V \subset L^{(p-1)/2}(\Omega) \).

Then we suppose \( a \in L^p(\Omega) \), \( b_i \), \( d_i \in L^q(\Omega) \) \((i = 1, \ldots, n)\), \( a \) positive constant, \( f_i \in L^2(\Omega) \), \( b_i \in L^2(\Omega) \), \( a \in L^q(\Omega) \), \( d_i \in L^q(\Omega) \)

\[ a = \sum_{i=1}^{n} (d_i a)_{i} \geq 0 \text{ in the sense of distributions,} \]

\[ a (u, v) = \int_{\Omega} \left( \sum_{i=1}^{n} a_{i} u_{i} v_{i} + \sum_{i=1}^{n} (b_{i} u_{i} v_{i} + d_{i} u_{i} v_{i}) \right) \, dx. \]

The result proven in theorem 1 yields the following

**Corollary 1.** Suppose that the hypotheses mentioned above are satisfied. Then there exists a solution \( u \in V \) of the Dirichlet problem

\[ \left\{ \begin{array}{l}
 u (u, v) = \sum_{i=1}^{n} \int_{\Omega} f_{i} u_{i} \, dx \\
 u \in V.
\end{array} \right. \]

**PROOF.** Let \( S_{\Omega} \) be the ball

\[ S_{\Omega} = \{ x : x \in \mathbb{R}^n, |x| < m \}. \]
and $u_m$ the solution of the Dirichlet problem

$$\begin{aligned}
\left\{ \begin{array}{l}
\Delta u_m(x) = 0 \\
\quad \forall \, \psi \in H^1_0(D \cap S_m)
\end{array} \right.
\end{aligned}$$

$$u_m \in H^1_0(D \cap S_m) \quad (m = 1, 2, \ldots).$$

From the hypotheses and theorem 1 we get

$$\| (u_m) \|_{L^p(D \cap S_m)} \leq K \sum_{i=1}^n \| f_i \|_{L^p(D \cap S_m)} \quad (m = 1, 2, \ldots)$$

where the constant $K$ depends only on $n, \nu, \sum \| b_i \|_{L^p(D)}$ and not on $m$. Since $H^1_0(D \cap S_m) \subset V$ for any $m \in \mathbb{N}$, from (24) it follows

$$\| u_m \|_V \leq K \sum_{i=1}^n \| f_i \|_{L^p(D)} \quad (m = 1, 2, \ldots)$$

From (25) there exists a subsequence, extracted from $\{ u_m \}_{m \in \mathbb{N}}$, weakly convergent in $V$ to a function $u$ which is a solution of problem (22), as it can easily be verified through (23).

**REFERENCES**


**SUMMARY.** In this paper we consider a solution of the Thren (0.5) interpolation problem given by piecewise polynomial functions.

**Introduzione.**

J. Sorayo e P. Turner hanno studiato, in un loro ben noto articolo [1], il problema dell'esistenza e dell'unicità di un polinomio interpolante, di grado non superiore a $(2n - 1)$, che in $n$ punti assegnati, assume valori assegnati. Negli stessi punti sono dati anche i valori della derivata seconda del polinomio. Per una esauriente bibliografia sull'argomento di interpolazione, vedi interpolazione lacunare di tipo (0, 2), si veda, per esempio, [2].

Nella presente nota viene studiato il problema dell'interpolazione (0, 2) mediante funzioni polinomiali a tratti.

1. Assegnato, nell'intervallo chiuso $I = [a, b]$, il sistema di punti

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b, \quad N > 3,$$

che dà luogo agli intervalli $I_k = [x_{k-1}, x_{k+1}]$, di ampiezza

$$h_{k, k+1} = h_k = x_{k+1} - x_k, \quad k = 0, 1, \ldots, N - 1.$$

(*) Estratto da Redazione il 5-11-1971.

(**) Il lavoro, perfezionato a Trani durante un soggiorno dell'A. presso la Università di Alberta, Edmonton (Alberta), è stato presentato al IX convegno dell'Unione Matematica Italiana, tenutosi a Bari dal 37-X al 3-X 1971.