Generalized Maximum Principle for Divergence Form Elliptic Equations in Unbounded Domains

Dedicated to the memory of prof. Guido Stampacchia.

Abstract. – *In this note I extend some previuos results concerning a generalized maximum principle for linear second order elliptic equations in divergence form, to the case of unbounded domains.*

1. – Introduction

In two previous works ([1], [2]) I have studied a generalized maximum principle for linear second order elliptic partial differential equations in divergence form and in bounded domains. In particular I have proved that if there exists a positive supersolution w in Ω, then every supersolution non negative on $\partial\Omega$ is also non negative in Ω, and conversely.

The aim of the present note is to extend, at least partially, these results to the case in which the domain Ω in $\mathbb{R}^n$ is unbounded. In this situation the complete continuity of the immersion of $H^1(\Omega)$ in $L^2(\Omega)$ is no longer true, so that many of the proofs already used in [1], [2] must be completely changed.

2. – Notations and hypotheses

Let Ω be an open connected subset of $\mathbb{R}^n$, not necessarily bounded (for simplicity we suppose $n \geq 3$, although the results could be easily extended to the case $n = 2$). We refer, for example, to [5], [8] for the definition of the spaces $H^{1,p}(\Omega)$, $H_o^{1,p}(\Omega)$; in $H^1(\Omega) := H^{1,2}(\Omega)$ we put, by definition,

$$||u_x||^2_{L^2(\Omega)} := \sum_{j=0}^{n} ||u_{x_j}||^2_{L^2(\Omega)}$$

where we assume as a norm, for instance, the quantity

$$||u||_{H^1(\Omega)} := \left\{ ||u||^2_{L^2(\Omega)} + \sum_{j=0}^{n} ||u_{x_j}||^2_{L^2(\Omega)} \right\}^{1/2}$$

Definition 1. *Let $p \geq 1,\ \delta > 0,\ f \in L^p_{loc}(\Omega)$; we define*

$$\omega(f,p,\delta) := \sup\{||f||_{L^p(E)}:\ E \text{ measurable},\ E \subset \Omega,\ \text{meas } E \leq \delta\}$$

$$X^p(\Omega) := \{f \in L^p_{loc}(\Omega):\ \omega(f,p,\delta) < +\infty\ \forall \delta > 0\}$$

$$X^p_o(\Omega) := \{f \in X^p(\Omega):\ \lim_{\delta \to 0^+} \omega(f,p,\delta) = 0\ \}$$

For further properties of these spaces see [3].

Suppose now $a_{ij} \in L^\infty(\Omega)$ $(i,j = 1,2,\dots n)$, $\sum_{i,j=1}^n a_{ij}t_it_j \geq \nu|t|^2\ \forall t \in \mathbb{R}^n$, with ν a positive constant; $b_i,\ d_i \in X^p(\Omega),\ p > n\ (i = 1,2,\dots,n),\ c \in X^{p/2}(\Omega)$. Then we define

$$a(u,v) := \int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n}(b_iu_{x_i}v + d_iuv_{x_i}) + cuv \right\} dx$$

We note that this expression, for the hypotheses on the coefficients and Theorem 1 of [3], is a bilinear form on $H^1_o(\Omega) \times H^1_o(\Omega)$.

3. – Preliminary lemmata

Lemma 1. *Suppose $w \in H^1_{loc}(\Omega)$ such that $w_x \in X^n(\Omega)$ and* ess $\inf_\Omega w > 0$. *If $u \in H^1_o(\Omega)$ it turns out $u/w \in H^1_o(\Omega)$ and*

$$||u/w||_{H^1(\Omega)} \leq K_1||u||_{H^1(\Omega)} \tag{1}$$

where K_1 is a constant depending on n, ess $\inf_\Omega w$ *and $\omega(w_x, n, 1)$.*

Proof. It is not a restriction to suppose $u \in C^1_o(\Omega)$ since this space is dense in $H^1_o(\Omega)$ by definition (provided the constant K_1 does not depend on the support of u). Let Q be a cube in $\mathbb{R}^n$, with side length 1. First of all we have trivially

$$||u/w||_{L^2(\Omega\cap Q)} \leq (\text{ess inf}_\Omega w)^{-1}||u||_{L^2(\Omega\cap Q)} \tag{2}$$

As what concerns the derivatives, it turns out

$$(u/w)_{x_i} = u_{x_i}/w - uw_{x_i}/w^2$$

and therefore

$$\begin{aligned} &||(u/w)_x||_{L^2(\Omega\cap Q)} \leq \\ &\quad \leq (\text{ess inf}_\Omega w)^{-1}||u_x||_{L^2(\Omega\cap Q)} + (\text{ess inf}_\Omega w)^{-2}||uw_x||_{L^2(\Omega\cap Q)} \end{aligned} \tag{3}$$

We now use Hölder and Sobolev inequalities (in the form of Lemma 2 of [4])

$$\begin{aligned} &||uw_x||^2_{L^2(\Omega\cap Q)} \leq ||u||^2_{L^{2^*}(\Omega\cap Q)}||w_x||^2_{L^n(\Omega\cap Q)} \leq \\ &\leq 2K_2\big[||u||^2_{L^2(\Omega\cap Q)} + ||u_x||^2_{L^2(\Omega\cap Q)}\big]||w_x||^2_{L^n(\Omega\cap Q)} \end{aligned} \tag{4}$$

where $2^* := 2n/(n-2)$ and K_2 is the constant of Lemma 2 of [4] (which depends only on n).

Let us consider now a family of cubes $\{Q_j\}_{j\in\mathbb{N}}$ with side length 1 such that $Q_i \cap Q_j = \emptyset$ when $i \neq j$ and $\cup_{j=1}^{+\infty}\overline{Q_j} = \mathbb{R}^n$. Let us rewrite (2) by replacing Q by Q_j and sum with respect to j (the function u can be defined equal to zero outside Ω). By remembering that by hypothesis it is $w_x \in X^n(\Omega)$, we get

$$||uw_x||^2_{L^2(\Omega)} \leq 2K_2\omega(w_x, n, 1)\left[||u||^2_{L^2(\Omega} + ||u_x||^2_{L^2(\Omega)}\right] \tag{5}$$

From (2), (3), (5) we easily reach the assertion (1). $\square$

The following lemma may be understood as a partial extension of Theorem 1 of [1] to the case of unbounded domains; the proof also is similar but it must be adapted to the new situation.

Lemma 2. *Suppose that the hypotheses listed in Section 2 are verified, and furthermore: there exists a function $w \in L^\infty(\Omega) \cap H^1_{loc}(\Omega)$ such that* ess $\inf_\Omega w > 0$, $w_x \in X^2(\Omega)$, *and w is a solution of the inequality $a(w, v) \geq 0$ $\forall v \in H^1_o(\Omega)$, $v \geq 0$ in Ω. Then if $u \in H^1(\Omega)$ is such that $u \leq 0$ on $\partial\Omega$ in the sense of $H^1(\Omega)$ and $a(u, v) \leq 0$ $\forall v \in H^1_o(\Omega)$, $v \geq 0$, it turns out $u \leq 0$ in Ω.*

Proof. It is not a restriction to suppose, for simplicity, that ess $\inf_\Omega w = 1$. In order to reach the conclusion, suppose by contradiction that $m :=$ ess $\sup_\Omega u > 0$. Since $w \in L^\infty(\Omega)$ by hypothesis, for any $k > 0$ sufficiently small it is ess $\sup_\Omega(u - kw) > 0$. Define now

$$k_o := \sup\{k \in \mathbb{R} : \text{ ess } \sup_\Omega(u - kw) > 0\}$$

I state that

$$\lim_{k\to k_o^-} \text{meas}\{x \in \Omega : \; u(x) - kw(x) > 0\} = 0 \tag{6}$$

This is obvious if $k_o = +\infty$; if $k_o \in \mathbb{R}$ it turns out

$$\lim_{k\to k_o^-} \text{meas}\{x \in \Omega : \; u(x) - kw(x) > 0\} = $$
$$= \text{meas}\{x \in \Omega : \; u(x) - k_o w(x) = 0\} \tag{7}$$

(In fact note that, by definition of k_o, it is meas$\{x \in \Omega : \; u(x) - kw(x) > 0\} = 0$ if $k > k_o$). But the function $u - k_o w$ is solution of the inequality

$$a(u - k_o w, v) \leq 0 \; \forall v \in H^1_o(\Omega), \; v \geq 0 \text{ in } \Omega$$

If it were meas$\{x \in \Omega : \; u(x) - kw(x) = 0\} > 0$, since it is also clearly $u(x) - k_o w(x) \leq 0$ a.e. in Ω, we should have $u - k_o w = 0$ in Ω by Corollary 1 of [1] (clearly valid also for unbounded domains). This is impossible since $w \notin H^1_o(\Omega)$, therefore (7) and then (6) are proved.

We now want to use $\max\{u - kw, 0\}$ as a test function, with $0 \leq k \leq k_o$, therefore we need to prove that this (non negative) function belongs to $H^1_o(\Omega)$. For simplicity we consider only the case $k = 1$, i. e. we prove that $\max\{u - w, 0\} \in H^1_o(\Omega)$ (this is not a restriction). Define $u^+ := \max\{u, 0\}$; since by hypothesis $u \in H^1(\Omega)$ and $u \leq 0$ on $\partial\Omega$ in the sense of $H^1(\Omega)$, it is easy to verify that $u^+ \in H^1_o(\Omega)$. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $C^1_o(\Omega)$ such that $\lim_j ||u^+ - u_j||_{H^1(\Omega)} = 0$ and define $\overline{u_j} := \max\{u_j - w, 0\}$; since by hypothesis $w \in H^1_{loc}(\Omega)$, we have $\overline{u_j} \in H^1_o(\Omega)(j = 1, 2, \dots)$. Define $A_j := \{x \in \Omega : \ u_j(x) > 1\}$, it turns out $\overline{u_j}(x) = 0$ in $\Omega \backslash A_j$ (since $w > 1$ in Ω), therefore

$$||(\overline{u_j})_x||_{L^2(\Omega)} \leq ||(u_j)_x||_{L^2(\Omega)} + ||w_x||_{L^2(A_j)} \leq \\ \leq ||(u_j)_x||_{L^2(\Omega)} + \omega(w_x, 2, \mathrm{meas}A_j) \tag{8}$$

and also trivially

$$||\overline{u_j}||_{L^2(\Omega)} \leq ||u_j||_{L^2(\Omega)} \ (j = 1, 2, \dots) \tag{9}$$

Furthermore, since

$$\max\{u - w, 0\} = \max\{u^+ - w, 0\} = \lim_j \overline{u_j} \quad \text{a.e. in } \Omega$$

we deduce also

$$\lim_j \mathrm{meas}A_j = \mathrm{meas}\{x \in \Omega : \ u(x) > 1\} < +\infty \tag{10}$$

From (8), (9), (10) we get that the sequence $\{\overline{u_j}\}_{j\in\mathbb{N}}$ is bounded in $H^1_o(\Omega)$; from known results a sequence of convex means of functions chosen from $\{\overline{u_j}\}_{j\in\mathbb{N}}$ converges strongly in $H^1_o(\Omega)$. This proves that $\max\{u - w, 0\} \in H^1_o(\Omega)$.

By the same proof we may verify that

$$\max\{u - kw, 0\} \in H^1_o(\Omega) \ \forall k > 0 \tag{11}$$

Now define, for brevity, $u_k := \max\{u - kw, 0\}$. We can choose this function u_k as the test function v in the inequality

$$a(u - kw, v) \leq 0 \ \forall v \in H^1_o(\Omega), \ v \geq 0$$

obtaining

$$a(u_k, u_k) \leq 0 \ \forall k > 0 \tag{12}$$

At this point we can proceed as in [1], Theorem 1. From (6), when $k < k_o$ is sufficiently near to k_o, the measure of $\{x \in \Omega : \ u_k(x) > 0\}$ is arbitrarily small. Taking into account the hypotheses made on the coefficients $a_{ij}, \ b_i, \ d_i, \ c$ of $a(.,.)$, we can find some $k < k_o$ such that (from (12)) $u_k = 0$ a.e. in Ω, a contradiction. □

4. – Main result

Theorem 1. *Suppose that the hypotheses listed in Section 2 are verified, and furthermore: there exists a function $w \in L^\infty(\Omega) \cap H^1_{loc}(\Omega)$ such that* ess $\inf_\Omega w > 0$, $w_x \in$ *$X^p(\Omega)$ with $p > n$, and w is a solution of the inequality $a(w, v) \geq \int_\Omega v \, dx \ \forall v \in$*

$H^1_o(\Omega),\ v \geq 0$ in Ω. Then for any $T \in H^{-1}(\Omega)$ there exists one and only one solution u of the Dirichlet problem

$$\begin{cases} a(u,v) =< T, v >_{H^1_o(\Omega)} \ \ \forall v \in H^1_o(\Omega), \\ u \in H^1_o(\Omega) \end{cases} \tag{13}$$

and there exists a constant K_3, depending on the coefficients of $a(.,.)$, n, Ω but not depending on T, u, such that

$$||u||_{H^1(\Omega)} \leq K_3 ||T||_{H^{-1}(\Omega)} \tag{14}$$

Proof. It is evidently sufficient to prove that the a priori inequality (14) is valid for the solution u of the Dirichlet problem (13). For what proved in [4] (Lemma 4), it is sufficient to prove (14) in the particular case in which $< T, v >:= \int_\Omega f v \, dx$ with $f \in H^1_o(\Omega)$ or, more generally, $f \in L^2(\Omega)$. Therefore let u be the solution of the Dirichlet problem

$$\begin{cases} a(u,v) = \int_\Omega f v \, dx \ \forall v \in H^1_o(\Omega), \\ u \in H^1_o(\Omega) \end{cases} \tag{15}$$

where f is a given function in $L^2(\Omega)$; we need to prove the existence of a constant K_3 such that the a priori inequality

$$||u||_{L^2(\Omega)} \leq K_3 ||f||_{L^2(\Omega)} \tag{16}$$

is valid (this is sufficient as in [4]).

Given $f \in L^2(\Omega)$, we can write $f = \max\{f, 0\} + \min\{f, 0\}$. If we denote by $u_1,\ u_2$ the solutions of the Dirichlet problems

$$\begin{cases} a(u_1,v) = \int_\Omega \max\{f,0\} v \, dx \ \forall v \in H^1_o(\Omega), \\ u_1 \in H^1_o(\Omega) \end{cases} \tag{17}$$

$$\begin{cases} a(u_2,v) = \int_\Omega \min\{f,0\} v \, dx \ \forall v \in H^1_o(\Omega), \\ u_2 \in H^1_o(\Omega) \end{cases} \tag{18}$$

we have, for the uniqueness of the solution (Lemma 2), $u = u_1 + u_2$. Therefore it is sufficient to prove inequlities of the type

$$||u_1||_{L^2(\Omega)} \leq K_3 || \max\{f, 0\}||_{L^2(\Omega)} \tag{19}$$

$$||u_2||_{L^2(\Omega)} \leq K_3 || \min\{f, 0\}||_{L^2(\Omega)} \tag{20}$$

in order to reach (16). By proceeding in this way in conclusion it is not a restriction to suppose, in order to prove (16), that $f \geq$ in Ω.

To this end, let z be the solution of the Dirichlet problem

$$\begin{cases} \displaystyle\int_\Omega \{w \sum_{i,j=1}^n a_{ij} z_{x_i} \phi_{x_j} + w \sum_{i=1}^n (b_i - d_i) z_{x_i} \phi - \\ \displaystyle - \sum_{i,j=1}^n a_{ij} w_{x_i} z_{x_j} \phi + z\phi\} \, dx = \int_\Omega f \phi \, dx \ \forall \phi \in H^1_o(\Omega) \\ z \in H^1_o(\Omega) \end{cases} \tag{21}$$

We remark that, for the hypotheses made on the function w and on the coefficients $a_{ij},\ b_i,\ d_i$, the Dirichlet problem (21) satisfies the hypotheses of Theorem 1 of [4], therefore there exists one and only one solution z of problem (21) and it turns out

$$||z||_{L^2(\Omega)} \leq K_3||f||_{L^2(\Omega)} \tag{22}$$

where the constant K_3 depends only on the coefficients of $a(.,.)$, n and Ω. Furthermore, since we have supposed $f \geq 0$ in Ω, it is also $z \geq 0$ in Ω (Lemma 1 of [4]).

Now we follow a procedure already used in [7], [6] for elliptic equations in non divergence form, i. e. the use of the function u/w as a solution of another equation. In fact we have

$$\int_\Omega \{w \sum_{i,j=1}^n a_{ij}(u/w)_{x_i}\phi_{x_j} + w\sum_{i=1}^n (b_i - d_i)(u/w)_{x_i}\phi -$$
$$- \sum_{i,j=1}^n a_{ij}w_{x_i}(u/w)_{x_j}\phi\}\, dx + a(w, u\phi/w) = a(u,\phi)\ \forall\phi \in H_o^1(\Omega) \tag{23}$$

This equation can be proved by a simple calculation (recall that $u/w \in H_o^1(\Omega)$ for our hypotheses and Lemma 1). By hypothesis we have also

$$a(u,v) = \int_\Omega fv\, dx\ \forall v \in H_o^1(\Omega) \tag{24}$$

$$a(w,v) \geq \int_\Omega v\, dx\ \forall v \in H_o^1(\Omega),\ v \geq 0 \tag{25}$$

therefore from (21), (23), (24), (25) we deduce

$$\int_\Omega \{w \sum_{i,j=1}^n a_{ij}(z - u/w)_{x_i}\phi_{x_j} + w\sum_{i=1}^n (b_i - d_i)(z - u/w)_{x_i}\phi -$$
$$- \sum_{i,j=1}^n a_{ij}w_{x_i}(z - u/w)_{x_j}\phi\}\, dx \geq 0\ \forall\phi \in H_o^1(\Omega),\ \phi \geq 0 \tag{26}$$

From (26) and Lemma 1 of [4], it follows

$$u/w \leq z \ \text{ a.e.} \in \Omega \tag{27}$$

But it is also, for the same Lemma, $u \geq 0$ a.e. in Ω, so from (27) we get easily

$$||u||_{L^2(\Omega)} \leq ||w||_{L^\infty(\Omega)}||z||_{L^2(\Omega)} \tag{28}$$

from which and (22) the conclusion (16) is attained. □

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