J Optim Theory Appl
DOI 10.1007/s10957-015-0744-6

Existence of Optimal Points Via Improvement Sets

Maurizio Chicco[1] · **Anna Rossi**[1]

Received: 9 October 2014 / Accepted: 20 April 2015

Abstract According to the recent definition of efficiency via improvement sets, the aim of this paper is to characterize the set of optimal points for a set. New existence results are proved in multicriteria situations, and their novelty is illustrated via several examples. Moreover, the study of an economic model is provided as an example of application of our achievements.

Keywords Vector optimization · Improvement sets · Optimal points · Existence results

Mathematics Subject Classification 90C26 · 90C29

1 Introduction

Most real-life problems are subject to decisions that must be taken according to appropriate optimality criteria. The original concepts of proper efficiency, proposed by Pareto in a pioneering paper, and approximate proper efficiency were later on modified and formulated in a more general framework by many authors.

The unification of the various definitions of efficiency has recently received a special attention in the literature on vector optimization. To our knowledge, the most general definition is reported in [1]. From the economic point of view, the research of such

✉ Anna Rossi
rossi@dime.unige.it

Maurizio Chicco
chicco@dime.unige.it

1 DIME, Scuola Politecnica, Università di Genova, Piazzale Kennedy, pad. D, 16129 Genoa, Italy

Published online: 06 May 2015

points is very general since the preference relation can be given on a Banach space and determined by a utility function or/and on a topological vector space by a order set, which is not necessarily a cone. For other details about mathematical economics, see [2] and references therein. When a convex cone is given, by specializing the order set, it is possible to recover the classical definition of efficient, weak efficient, strict efficient, and Henig proper efficient solutions (see [3]). There exist other attempts to unify the definition of efficiency. One of these is due to Ha, who in [4] introduced the notion of minimal point of a set with respect to a proper open (not necessarily convex) cone with apex at the origin. We point out that the definition given by Ha preceded the one by Flores-Bazàn and Hernàndez [1], but it can be deduced from the latter.

Another attempt to unify was carried out first in [5], where the concept of efficiency for maximization in a finite dimensional setting is presented, then it has been generalized to a real locally convex Hausdorff topological vector space by Gutiérrez et al. in [6]. It is based on special sets called improvement sets, which have two properties: the exclusion property and the comprehensive property. We observe that the definition of efficiency, given in [5], follows from the one quoted in [1] with suitable choices of the function and of the order set.

In order to formulate a definition that unifies the concepts of efficiency, in most of the papers mentioned so far, it is possible to find optimality conditions for vector optimization problems, carried out via scalarizations and via approximate subdifferentials in the convex case or via the Mordukhovich subdifferential, when nonconvexity is assumed (see e.g., [1,3,7,8] and references therein), and existence results for efficient points via topological properties of the sets (see e.g., [5,9,10]).

Inspired by such results, in this paper, we study the existence of efficient points via topological properties of the sets, currently in a finite dimensional setting. To emphasize the importance of the topic, we quote the recent paper [11], that in this framework applies the equilibria concept to study multicriteria games.

The outline of this article is as follows. In Sect. 2, we recall the definition of improvement set and some preliminary results. Section 3 introduces the characterization of the improvement sets "close" to zero. In Sect. 4, we present some conditions (necessary, sufficient or both) for the existence of optimal points, giving several examples to illustrate our results and their novelty. Moreover, we provide an example, which shows that our results can be applied to an economic model. Section 5 provides an answer to a question proposed by Zhao and Yang in [12]. Finally, some research topics and the main conclusions are presented in Sects. 6 and 7, respectively.

2 Definitions and Preliminary Results

We write $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and denote by $\mathbf{e}_i$ the vector of $\mathbb{R}^n$ with the i-th component equal to 1 and the others 0, so that, for every $\mathbf{x} \in \mathbb{R}^n$ with components $x_1, x_2, \ldots, x_n$, it turns out $\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i$. In case we have a sequence $\{\mathbf{x}_m\}_{m\in\mathbb{N}} \subset \mathbb{R}^n$, we denote by $(x_m)_i$ the i-th component of the vector $\mathbf{x}_m$, in such a way that $\mathbf{x}_m = \sum\limits_{i=1}^{n} (x_m)_i \mathbf{e}_i \; (m \in \mathbb{N})$.

By $\mathbb{R}^n_+$, we mean the points in $\mathbb{R}^n$ with all coordinates positive or null, by $\mathbb{R}^n_{++}$ those with all coordinates strictly positive (analogously for $\mathbb{R}^n_-$ and $\mathbb{R}^n_{--}$) and, given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, for every $i = 1, \dots, n$: $\mathbf{x} \geqq \mathbf{y} \Leftrightarrow x_i \geq y_i$; $\mathbf{x} > \mathbf{y} \Leftrightarrow x_i > y_i$ (analogous definitions for $\leqq$, $<$).
For $A \subset \mathbb{R}^n$: $int(A)$ and $bd(A)$ are the interior and the boundary of A, respectively; $cl(A)$ is the closure of A; $d(\mathbf{x}, A) = \inf\{d(\mathbf{x}, \mathbf{a}) : \mathbf{a} \in A\}$, where $d(\mathbf{x}, \mathbf{a})$ is the usual distance in $\mathbb{R}^n$, i.e., $d(\mathbf{x}, \mathbf{a}) := \{\sum_{i=1}^n (x_i - a_i)^2\}^{1/2}$.

We say that $V \subset \mathbb{R}^n$ is *upper-bounded* iff there exists $\mathbf{b} \in \mathbb{R}^n$ such that $\mathbf{x} \leqq \mathbf{b}\ \forall \mathbf{x} \in V$.

In the following, we will write $\langle \mathbf{a}, \mathbf{b} \rangle$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, to indicate the internal (scalar) product of two vectors, i.e., $\langle \mathbf{a}, \mathbf{b} \rangle := \sum_{i=1}^n a_i b_i$.

Definition 2.1 Let $A \subset \mathbb{R}^n$. We define the upper comprehensive set of A as follows: $u\text{-}compr(A) := \{\mathbf{x} \in \mathbb{R}^n : \text{ there exists } \mathbf{a} \in A \text{ s.t. } \mathbf{a} \leqq \mathbf{x}\}$

It is easy to prove:

Proposition 2.1 $u\text{-}compr(A) = \bigcup_{\mathbf{a} \in A} \left(\mathbf{a} + \mathbb{R}^n_+\right)$.

Definition 2.2 A subset E of $\mathbb{R}^n$ is called ”upper comprehensive set” iff $u\text{-}compr(E) = E$.

Definition 2.3 Let $E \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ be an upper comprehensive set. We shall call E an improvement set of $\mathbb{R}^n$, and we will denote by $\Im^n$ the family of the improvement sets in $\mathbb{R}^n$.

Remark 2.1 From the above definitions, it follows that $E \subset \mathbb{R}^n$ is an improvement set iff it has the two following properties:
i) $\mathbf{0} \notin E$; ii) if $\mathbf{x} \in E$ and $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} \geqq \mathbf{0}$, then $\mathbf{x} + \mathbf{y} \in E$.

3 Improvement Sets and E-Optimal Points

Proposition 3.1 (see [5] Proposition 3.1.) *The following relations hold:*

(i) $\Im^n$ *is a lattice, i.e., if* $E_1, E_2 \in \Im^n$*, then* $E_1 \bigcap E_2 \in \Im^n$, $E_1 \bigcup E_2 \in \Im^n$.
(ii) The largest improvement set is $E_o = \mathbb{R}^n \setminus \mathbb{R}^n_-$.
(iii) The smallest is $E_{oo} = \emptyset$.

Definition 3.1 Let A, $E \subset \mathbb{R}^n$, E improvement set. We say that $\mathbf{a} \in A$ is an E-optimal point iff $(\mathbf{a} + E) \cap A = \emptyset$ and denote this by writing $\mathbf{a} \in O^E(A)$.

Remark 3.1 In other words: if $A \subset \mathbb{R}^n$ and $E \in \Im^n$, we say that $\mathbf{a} \in O^E(A)$ iff $\mathbf{a} \in A$ and $\mathbf{a} + \mathbf{e} \notin A$, $\forall \mathbf{e} \in E$.

We get an equivalent definition of the set $O^E(A)$ by the following proposition (its easy proof is left to the reader).

Proposition 3.2 *Let* A, $E \subset \mathbb{R}^n$, E *improvement set. Then* $\mathbf{x} \in O^E(A)$ *iff* $\mathbf{x} \in A$ *and* $\mathbf{a} - \mathbf{x} \notin E\ \ \forall \mathbf{a} \in A$.

Several examples of E-optimal sets can be found in [5].

In Sect. 4, we will give conditions in order that $O^E(A)$ is not empty with the improvement set $E_+ := \mathbb{R}^n_{++} = \{\mathbf{x} \in \mathbb{R}^n : x_i > 0\ \forall i = 1, 2, \dots, n\}$, having zero distance from $\mathbf{0}$. For this reason now, we start by giving some properties related to improvement sets with such property.

Proposition 3.3 *Let $E \in \Im^n$ be such that $d(\mathbf{0}, E) = 0$. Then $E_+ \subset E$.*

Proof Let $\mathbf{x} \in E_+$ and $\delta := \min\{x_i : i = 1, 2, \dots, n\} > 0$ by the definition of E_+. Since $d(\mathbf{0}, E) = 0$, there exists $\mathbf{y} \in E$ such that $|\mathbf{y}| = d(\mathbf{0}, \mathbf{y}) < \delta$. Now let $\hat{\mathbf{y}} \in \mathbb{R}^n$ with components defined as $\hat{y}_i = \max\{y_i, 0\}$ $(i = 1, 2, \dots, n)$.

By definition of improvement set it turns out $\hat{\mathbf{y}} \in E$ and $|\hat{\mathbf{y}}| \le |\mathbf{y}| < \delta$. Then, for every $i = 1, 2, \dots, n$, we have $\hat{y}_i < \delta \le x_i$, so that $\hat{y}_i < x_i$.

Since $\hat{\mathbf{y}} \in E$ and $\mathbf{x} > \hat{\mathbf{y}}$, thanks to Remark 2.1, we have $\mathbf{x} \in E$. □

Proposition 3.4 *Let $E \in \Im^n$ be such that $d(\mathbf{0}, E) = \delta \ge 0$. Then*

$$O^E(A) \subset \{\mathbf{x} \in A : d(\mathbf{x}, bd(A)) \le \delta\}.$$

Proof We prove that if $\mathbf{x} \in A$ and $d(\mathbf{x}, bd(A)) > \delta$, then $\mathbf{x} \notin O^E(A)$. Let $\mathbf{x} \in A$ be such that $d(\mathbf{x}, bd(A)) > \delta$ and $\mathbf{e} \in E$ such that $\delta \le d(\mathbf{0}, \mathbf{e}) < d(\mathbf{x}, bd(A))$. Such $\mathbf{e}$ exists since $d(\mathbf{0}, E) = \delta$, $d(\mathbf{x}, bd(A)) > \delta$ and by definition of distance. Therefore, we obtain $d(\mathbf{x}, \mathbf{x} + \mathbf{e}) = d(\mathbf{0}, \mathbf{e}) < d(\mathbf{x}, bd(A))$, which proves that $\mathbf{x} + \mathbf{e} \in A$ and $\mathbf{x} \notin O^E(A)$. □

Corollary 3.1 *If in the previous Proposition 3.4 we consider $\delta = 0$, we have:*

1. *$O^E(A) \subset bd(A)$;*
2. *if A is an open set, $O^E(A) = \emptyset$.*
   (In this context see also [11], Proposition 11.1).

Proposition 3.5 *Let A, $E \subset \mathbb{R}^n$, E improvement set; suppose A closed and E open. Then $O^E(A)$ is closed.*

Proof Let $\mathbf{x}_k \in O^E(A)$ $(k = 1, 2, \dots)$ such that $\lim_k \mathbf{x}_k = \overline{\mathbf{x}}$. We must prove that $\overline{\mathbf{x}} \in O^E(A)$. First of all, we note that $\overline{\mathbf{x}} \in A$, since A is closed. By contradiction, suppose $\overline{\mathbf{x}} \notin O^E(A)$; then there exists $\overline{\mathbf{e}} \in E$ such that $\overline{\mathbf{x}} + \overline{\mathbf{e}} \in A$. This relation can be written, equivalently, $\mathbf{x}_k + \overline{\mathbf{e}} + (\overline{\mathbf{x}} - \mathbf{x}_k) \in A$. Since E is open by hypothesis, the point $\overline{\mathbf{e}}$ is internal to E, so $\overline{\mathbf{e}} + (\overline{\mathbf{x}} - \mathbf{x}_k) \in E$ when k is sufficiently large: a contradiction since $\mathbf{x}_k \in O^E(A)$. □

4 Existence of E-Optimal Points

Before we focus our attention on the case where the improvement set is E_+, let us consider next theorem, which holds for every improvement set $E \subset \mathbb{R}^n$.

Theorem 4.1 *Let $E \in \Im^n$ and A a closed and nonempty set of $\mathbb{R}^n$. Suppose that there exists $\mathbf{p} \in \mathbb{R}^n$ with $p_i > 0$ for every $i \in \{1, 2, \dots, n\}$ such that $\langle \mathbf{p}, \mathbf{e} \rangle > 0\ \forall\, \mathbf{e} \in E$ and A is upper-bounded. Then $O^E(A) \neq \emptyset$.*

Proof Since A is upper-bounded, we have $\mu := \sup\{\langle \mathbf{p}, \mathbf{a}\rangle : \mathbf{a} \in A\} < +\infty$. Therefore, there exists a sequence $\{\mathbf{x}_m\}_{m\in \boldsymbol{N}} \subset A$ such that

$$\mu - \frac{1}{m} \le \langle \mathbf{p}, \mathbf{x}_m\rangle \le \mu \tag{1}$$

The sequence $\{\mathbf{x}_m\}_{m\in \boldsymbol{N}}$ is bounded in $\mathbb{R}^n$. In fact, we can write (1) as

$$\mu - \frac{1}{m} \le \sum_{i\in I-} p_i (x_m)_i + \sum_{i\in I+} p_i (x_m)_i \le \mu \tag{2}$$

where $\sum_{i\in I-}$ means the sum on the indices i for which $(x_m)_i < 0$ and $\sum_{i\in I+}$ the sum on the indices i for which $(x_m)_i > 0$. From (2), we easily obtain

$$\sum_{i\in I-} p_i |(x_m)_i| \le \left|\mu - \frac{1}{m}\right| + \sum_{i\in I+} p_i (x_m)_i, \tag{3}$$

and recalling that A is supposed to be upper-bounded, we deduce that the sequence $\{\mathbf{x}_m\}_{m\in \boldsymbol{N}}$ is bounded in $\mathbb{R}^n$. Then $\{\mathbf{x}_m\}_{m\in \boldsymbol{N}}$ admits a subsequence $\{\mathbf{x}_{m_k}\}_{k\in \boldsymbol{N}}$ converging to $\bar{\mathbf{x}} \in A$, since A is closed. We have $\lim_{k\to\infty} \langle \mathbf{p}, \mathbf{x}_{m_k}\rangle = \langle \mathbf{p}, \bar{\mathbf{x}}\rangle = \mu$.

Now, if $\mathbf{e} \in E$, we get $\langle \mathbf{p}, \bar{\mathbf{x}} + \mathbf{e}\rangle = \langle \mathbf{p}, \bar{\mathbf{x}}\rangle + \langle \mathbf{p}, \mathbf{e}\rangle > \langle \mathbf{p}, \bar{\mathbf{x}}\rangle$, so that $\bar{\mathbf{x}} + \mathbf{e} \notin A$, $\bar{\mathbf{x}} \in O^E(A)$ and $O^E(A) \neq \emptyset$. □

Remark 4.1 (i) In Theorem 4.1, the assumption that $\mathbf{p} \in \mathbb{R}^n$ is such that $\langle \mathbf{p}, \mathbf{e}\rangle > 0\, \forall\, \mathbf{e} \in E$ implies $p_i \ge 0\ \forall\, i = 1, 2, \ldots, n$. In fact, if there were $\bar{i} \in \{1, 2, \ldots, n\}$ for which $p_{\bar{i}} < 0$ (we can take for simplicity $\bar{i} = 1$), by Remark 2.1, if $\mathbf{e} \in E$ and $\mathbf{y} = (y_1, 0, \ldots, 0)$ with $y_1 > 0$, $-p_1\, y_1 > \sum_{i=1}^{n} p_i\, e_i$, we have $\mathbf{y} \in \mathrm{cl}(E_+)$, so $\mathbf{e}+\mathbf{y} \in E$ and $\langle \mathbf{p}, \mathbf{e}+\mathbf{y}\rangle < 0$, in contradiction with the assumption.

(ii) If there exists no $\mathbf{p} \in \mathbb{R}^n$ such that $\langle \mathbf{p}, \mathbf{e}\rangle > 0$ for every $\mathbf{e} \in E$, then $O^E(A)$ can be empty. In fact let $A = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 \le 1\}$ and $E = \{(x, y) \in \mathbb{R}^2 : x > \min(0, -y)\}$. Clearly for every $(x, y) \in A$ it turns out $((x, y) + E) \cap A \neq \emptyset$.

(iii) If $\mathbf{x} \in A$ is such that there exists $\bar{i} \in \{1, 2, \ldots, n\}$ for which $\sup x_{\bar{i}} = +\infty$, then $O^E(A)$ may be empty. Let $A = \left\{(x, y) \in \mathbb{R}^2 : -2 \le x < 1,\, y \ge -\frac{1}{x-1}\right\}$ and $E = \{(x, y) \in \mathbb{R}^2 : x > 0,\, xy \ge 1\}$. The existence of $\mathbf{p} \in \mathbb{R}^n \setminus \{(0, 0)\}$ verifying the hypothesis of Theorem 4.1 is ensured, but for every $(x, y) \in A$, we have $((x, y) + E) \cap A \neq \emptyset$.

(iv) If all the components of $\mathbf{p}$ are not strictly positive, then $O^E(A)$ can be empty. Let $E = \{(x, y) \in \mathbb{R}^2 : x > 0\}$, $A = \{(x, y) \in \mathbb{R}^2 : x < 0,\, xy \ge 1\}$ and $\mathbf{p} = (1, 0)$. Then $O^E(A) = \emptyset$, even though $\langle \mathbf{p}, \mathbf{e}\rangle > 0\ \forall\, \mathbf{e} \in E$, A is closed and $\sup\{x_i : \mathbf{x} \in A\} < +\infty\ \forall\, i = 1, 2$.

(v) If in the previous theorem A is not closed, $O^E(A)$ can be empty. In fact, let $E = E_+$ and $A = \{(x, y) \in \mathbb{R}^2 : x < 0, xy > 1\}$, it turns out $O^E(A) = \emptyset$. (See also Corollary 3.1, 2))

(vi) Theorem 4.1 gives sufficient conditions to have $O^E(A) \neq \emptyset$ as Theorem 4.1 of [5], but, with the exception of the assumption that A is $\mathbf{p}$-upper-bounded, the other conditions are not comparable. We will see that with the following two examples, but let us first recall that if $\mathbf{p} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $E \in \Im^n$, then $\mathbf{p}$ is called a *separator* for E iff there exists a positive number t such that $\langle \mathbf{p}, \mathbf{e} \rangle > t$ for each $\mathbf{e} \in E$.

Let $E = E_+$ and $A = \{(x, y) \in \mathbb{R}^2 : x < 0,\ xy \geq 1\}$, E is not separable from $\{\mathbf{0}\}$, but if $\mathbf{p} = (1, 1)$, it follows $\langle \mathbf{p}, \mathbf{e} \rangle > 0\ \ \forall \mathbf{e} \in E$. So the hypotheses of Theorem 4.1 are verified, whereas those of Theorem 4.1 of [5] are not.

If we consider $E = \{(x, y) \in \mathbb{R}^2 : x \geq 2\}$, $A = \{(x, y) \in \mathbb{R}^2 : x < 0,\ xy > 1\}$ and $\mathbf{p} = (1, 0)$, then $\mathbf{p}$ is a separator of E and $\mathbf{0}$, but A is not closed, and there exists a component of $\mathbf{p}$ equal to 0. So the hypotheses of Theorem 4.1 in [5] are verified, while those of the above Theorem 4.1 are not.

Next example shows how the previous theorem can be applied to an economic model $\mathcal{E}$ as described in [13].

Example 4.1 Consider, as commodity space, $\mathbb{R}^n$ endowed with the usual norm topology, the only Hausdorff linear topology in $\mathbb{R}^n$.

Let $I = \{1, \ldots, m\}$ a finite number of consumers and, for each $i = 1, \ldots, m$, let $E_i := \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u} \geq \mathbf{a}_i,\ \mathbf{a}_i \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}\}$ the consumption set of the i-th consumer. We let $E := \bigtimes_{i=1}^m E_i$ and define the preference correspondences $P_i : E \rightrightarrows E_i$ as $P_i(\underline{\mathbf{u}}) = \mathbf{u}_i + E_i$, where $\underline{\mathbf{u}}$ will denote, now and in the following, an element of E. Moreover, let $\boldsymbol{\omega}_i$ the initial endowment of the i-th consumer, $\boldsymbol{\omega}_i \in E_i$, and $\boldsymbol{\omega} = \sum_{i \in I} \boldsymbol{\omega}_i$ the total endowment.

Hence we consider the pure exchange economy $\mathcal{E} = (\mathbb{R}^n, (E_i, P_i)_{i \in I}, (\boldsymbol{\omega}_i)_{i \in I})$ and define the set of feasible allocations for the economy $\mathcal{E}$ as

$$\mathcal{A}(\mathcal{E}) = \left\{ \underline{\mathbf{x}} \in E : \sum_{i \in I} \mathbf{x}_i \leq \boldsymbol{\omega} \right\}$$

Furthermore, if $\underline{\mathbf{u}}^* \in E$, let

$$P(\underline{\mathbf{u}}^*) := \{\underline{\mathbf{u}} = (\mathbf{u}_1, \ldots, \mathbf{u}_m) \in E : \mathbf{u}_i \in P_i(\underline{\mathbf{u}}^*), \quad i = 1, \ldots, m\}$$

The definition of weakly Pareto optimal allocation (see [13], Definition 3.1) is:

Definition 4.1 An allocation $\underline{\mathbf{u}}^* \in \mathcal{A}(\mathcal{E})$ is said to be weakly Pareto optimal for the economy $\mathcal{E}$ iff $\mathcal{A}(\mathcal{E}) \cap P(\underline{\mathbf{u}}^*) = \emptyset$.

Since $P(\underline{\mathbf{u}}^*) = \underline{\mathbf{u}}^* + E$, it turns out that $\underline{\mathbf{u}}^* \in E$ such that $\sum_{i \in I} \mathbf{x}_i \leq \boldsymbol{\omega}$ is a weakly Pareto optimal allocation for the economy $\mathcal{E}$ iff $u^* \in O^E\big(\mathcal{A}(\mathcal{E})\big)$. Then, it is sufficient to verify that Theorem 4.1 can be applied in this framework.

Clearly, E is an improvement set and $\mathcal{A}(\mathcal{E}) \neq \emptyset$, because $\underline{\boldsymbol{\omega}} \in \mathcal{A}(\mathcal{E})$, and it is closed in virtue of the closure of E. Moreover, since $\langle \underline{\mathbf{p}}, \underline{\mathbf{e}} \rangle = \sum_{k=1}^{n\,m} p_k\, e_k$, it is sufficient to choose $p_k = 1$ for every $k \in \{1, \ldots, n\,m\}$ to get $\langle \underline{\mathbf{p}}, \underline{\mathbf{e}} \rangle > 0$.

Finally, $\mathcal{A}(\mathcal{E}) \subset E$ is clearly upper- bounded.

From now on, we will focus our attention on the case in which the improvement set is E_+, so E-optimal points become weak Pareto efficient points with respect to $\mathbb{R}^n_{++}$.

There are many papers devoted to establish conditions implying the nonemptiness of the set of weak Pareto efficient points. Unfortunately, many of the conditions assumed imply the boundedness of such set. Here, we study the existence of weak Pareto efficient points of unbounded sets $A \subset \mathbb{R}^n$, giving the possibility that the set $O^{E_+}(A)$ is unbounded. In Remark 4.4, we will show that our achievements improve the results reported in the recent paper [14], where many results previously appeared in the literature are already encompassed and an improvement of Theorem 3.5 in [15] is presented.

We start with the two-dimensional case, for which we will give a necessary and sufficient condition ensuring that the set of the E-optimal points is not empty.

Theorem 4.2 *Let $A \subset \mathbb{R}^2$ a closed and nonempty set such that there exists $k_o \in \mathbb{R}$ for which $A \cap \{(x, y) \in \mathbb{R}^2 : x \geq k\} \neq \emptyset$ for all $k \geq k_o$.*

Moreover, let $I = \{x \in \mathbb{R} : \exists\, y \in \mathbb{R}$ such that $(x, y) \in A\}$ and, for $x \in I$, $\varphi(x) = \sup\{y \in \mathbb{R} : (x, y) \in A\}$. If, in addition, $\sup\{\varphi(x) : x \in I\} < +\infty$ and $\limsup_{x\to+\infty} \varphi(x) = \lambda \in \mathbb{R}$, then $O^{E_+}(A) \neq \emptyset$ iff there exists $\bar{x} \in I$ such that $\varphi(\bar{x}) \geq \lambda$.

Proof First we observe that, since $\sup\{\varphi(x) : x \in I\} < +\infty$, it follows $(x, \varphi(x)) \in bd(A) \subset A\ \forall x \in I$. Suppose there exists $(\bar{x}, \bar{y}) \in O^{E_+}(A)$; we prove that $\varphi(\bar{x}) \geq \lambda$.

In fact, on the contrary, suppose that $\varphi(\bar{x}) < \lambda$. In this case, by the definition of λ, there exists $\hat{x} \in I$, $\hat{x} > \bar{x}$ such that $\varphi(\hat{x}) > \varphi(\bar{x})$ and we can choose $e_1,\ e_2$ such that $0 < e_1 := \hat{x} - \bar{x},\ 0 < \varphi(\hat{x}) - \varphi(\bar{x}) \leq \varphi(\hat{x}) - \bar{y} := e_2$. Thus, $(e_1, e_2) \in E_+$, $(\hat{x}, \varphi(\hat{x})) = (\bar{x} + e_1, \bar{y} + e_2) \in A$, contrary to $(\bar{x}, \bar{y}) \in O^{E_+}(A)$.

Now we assume that there exists $\bar{x} \in I$ such that $\varphi(\bar{x}) \geq \lambda$ and let us first consider the case in which $\varphi(\bar{x}) > \lambda$. By definition of $\limsup_{x\to+\infty} \varphi(x) = \lambda \in \mathbb{R}$, there exists $\ell > 0$ such that, for every $x > \ell$, it follows $\varphi(x) < \varphi(\bar{x})$. Let $I_o = \{x \in I : x \geq \bar{x},\ \varphi(x) \geq \varphi(\bar{x})\}$. It turns out that $I_o \subset [\bar{x}, \ell]$. Now let $M = \sup\{\varphi(x) : x \in I_o\}$. There exists a sequence $\{x_m\}_{m\in N}$, $x_m \in I_o$ such that $\lim_m \varphi(x_m) = M$ and $(x_m, \varphi(x_m)) \in A$. If, up to a subsequence, $\lim_m x_m = x_o$, then $\lim_m (x_m, \varphi(x_m)) = (x_o, M) \in A$, since A is closed, so that $M = \varphi(x_o)$ and $\varphi(x) \leq \varphi(x_o)\ \forall x \in I_o$. Hence the point $(x_o, \varphi(x_o)) \in O^{E_+}(A)$ and the thesis is achieved.

If we assume that there exists $\bar{x} \in I$ such that $\varphi(\bar{x}) = \lambda$ and, for every $x \in I$, we have $\varphi(x) \leq \lambda$, it is immediate to establish that $(\bar{x}, \varphi(\bar{x})) \in O^{E_+}(A)$. □

Corollary 4.1 *In the hypotheses of the previous theorem, if E is an improvement set such that $d(\mathbf{0}, E) = 0$ and $O^E(A) \neq \emptyset$, then there exists $\bar{x} \in I$ such that $\varphi(\bar{x}) \geq \lambda$.*

By Proposition 3.3 $E_+ \subset E$, so $O^{E_+}(A) \supset O^E(A)$ thanks to Proposition 3.2, i) of [5]. Applying the necessary condition of Theorem 4.2, we may conclude.

Proposition 4.1 *In the hypotheses of Theorem 4.2, if instead we have that* $\limsup_{x\to+\infty} \varphi(x) = +\infty$*, then* $O^{E_+}(A) = \emptyset$.

Proof If we consider any point $(\bar{x}, \bar{y}) \in A$, by definition of lim sup, there exists a vector $(e_1, e_2) \in E_+$ such that $\varphi(\bar{x} + e_1) = \bar{y} + e_2$; hence $(\bar{x} + e_1, \bar{y} + e_2) \in A$ and $(\bar{x}, \bar{y}) \notin O^{E_+}(A)$. □

If there exists $\bar{k} \in \mathbb{R}$ such that $A \cap \{(x, y) \in \mathbb{R}^2 : x \geq \bar{k}\} = \emptyset$, we have

Proposition 4.2 *Let* $I = \{x \in \mathbb{R} : \exists\, y \in \mathbb{R}$ *for which* $(x, y) \in A\}$*, where* A *is a closed and nonempty set of* $\mathbb{R}^2$*. For* $x \in I$, $\varphi(x) = \sup\{y \in \mathbb{R} : (x, y) \in A\}$*. Moreover, we assume that* $\sup I = k_o \in \mathbb{R}$*.*

1. *If* $k_o \in I$*, then* $O^{E_+}(A) \neq \emptyset$*.*
2. *If* $k_o \notin I$ *and* $\limsup_{x\to k_o^-} \varphi(x) = +\infty$*, then* $O^{E_+}(A) = \emptyset$*.*
3. *If* $k_o \notin I$ *and* $\lim_{x\to k_o^-} \varphi(x) = -\infty$*, then* $O^{E_+}(A) \neq \emptyset$*.*

Proof Proof of 1): If $k_o \in I$, there exists $y_o \in \mathbb{R}$ such that $(k_o, y_o) \in A$ and, by definition of supremum, if $(e_1, e_2) \in E_+$, $(k_o + e_1, y_o + e_2) \notin A$, so that $(k_o, y_o) \in O^{E_+}(A)$.

Proof of 2): it is analogous to the one in Proposition 4.1, and therefore, we omit it.

Proof of 3): Let $x_1 \in I$ be such that $\mu := \sup\{\varphi(x) : x \in [x_1, k_o)\} < +\infty$ (such an x_1 exists, since $\lim_{x\to k_o^-} \varphi(x) = -\infty$ by hypothesis). Consider a sequence $\{x_m\}_{m\in N} \subset [x_1, k_o)$ such that $\lim_m \varphi(x_m) = \mu$; we can assume (eventually considering a subsequence) that there exists $\bar{x} := \lim_m x_m$. From the hypothesis $\lim_{x\to k_o^-} \varphi(x) = -\infty$, it follows $x_1 \leq \bar{x} < k_o$. Since A is closed, we have $(\bar{x}, \mu) \in A$; it is easy to verify that $(\bar{x}, \mu) \in O^{E_+}(A)$. □

Remark 4.2 In the previous proposition, we do not consider the case in which $k_o \notin I$ and $\limsup_{x\to k_o^-} \varphi(x) = \lambda \in \mathbb{R}$. In fact this case cannot happen, since $(k_o, \lambda) \in A$ in virtue of the closedness of A, therefore $k_o \in I$.

It is possible to extend Theorem 4.2 to sets $A \subset \mathbb{R}^n$ with $n > 2$ only in part. To this purpose, it is necessary to introduce some new notation.

If $\mathbf{x} \in \mathbb{R}^n$ $(n \geq 2)$, we consider $x' := (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, so that $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1}, x_n) = (x', x_n) \in \mathbb{R}^n$. Moreover, we denote by $||x'||$ the usual euclidean norm of $x' \in \mathbb{R}^{n-1}$, i. e. $||x'|| := \left\{\sum_{i=1}^{n-1} x_i^2\right\}^{1/2}$

Theorem 4.3 *Let* $A \subset \mathbb{R}^n$ $(n \geq 3)$ *a closed and nonempty set such that there exists* $k_o \in \mathbb{R}$ *for which* $A \cap cl(E_+) \cap \{(x', x_n) \in \mathbb{R}^n : \|x'\| > k\} \neq \emptyset$ *for every* $k \geq k_o$*. Moreover, let* $I := \{x' \in \mathbb{R}^{n-1} : \exists x_n \in \mathbb{R}\ s.\ t.\ (x', x_n) \in A \cap cl(E_+)\}$ *and, for* $x' \in I$, $\varphi(x') := \sup\{x_n \in \mathbb{R} : (x', x_n) \in A \cap cl(E_+)\}$*.*

If, in addition, $\sup\{\varphi(x') : x' \in I\} < +\infty$, $\limsup_{\|x'\|\to+\infty} \varphi(x') = \lambda \in \mathbb{R}$ *and there exists* $\bar{x}' \in I$ *such that* $\varphi(\bar{x}') \geq \lambda$*, then* $O^{E_+}(A) \neq \emptyset$*.*

Proof We can follow the one of Theorem 4.2. First, suppose there exists $\bar{x}' \in I$ such that $\varphi(\bar{x}') > \lambda$. From the hypothesis and the definition of lim sup, there exists $\ell > 0$ such that, for every $x' \in I$ with $\|x'\| > \ell$, it follows $\varphi(x') < \varphi(\bar{x}')$. Let $I_o = \{x' \in I : \|x'\| \geq \|\bar{x}'\|,\ \varphi(x') \geq \varphi(\bar{x}')\}$. It turns out that $\sup\{\|x'\| : x' \in I_o\} \leq \ell$. Now let $M = \sup\{\varphi(x') : x' \in I_o\}$. There exists a sequence $\{x'_m\}_{m \in N}$, $x'_m \in I_o$ such that $\lim_m \varphi(x'_m) = M$ and $(x'_m, \varphi(x_m)) \in A$.

If, up to a subsequence, $\lim_m x'_m = x'_o$, then $\lim_m (x_m, \varphi(x_m)) = (x'_o, M) \in A$, since A is closed, so that $M = \varphi(x'_o)$ and $\varphi(x') \leq \varphi(x'_o)\ \forall\, x' \in I_o$. Now we consider a vector $\mathbf{e} = (e', e_n) \in E_+$. It turns out $\|x'_o + e'\| > \|x'_o\|$, since $x'_o \in I$ and, if $x'_o + e' \notin I$, $(x'_o + e', \varphi(x'_o) + e_n) \notin A$; if $x'_o + e' \in I \setminus I_o$, $\varphi(x'_o + e') < \varphi(\bar{x}') < \varphi(x'_o) + e_n$, so, by definition of φ, $(x'_o + e', \varphi(x'_o) + e_n) \notin A$; if $x'_o + e' \in I_o$, by definition of supremum, $(x'_o + e', \varphi(x'_o) + e_n) \notin A$. Hence the point $(x'_o, \varphi(x'_o)) \in O^{E_+}(A)$ and the thesis follows.

In case there exists $\bar{x}' \in I$ such that $\varphi(\bar{x}) = \lambda$, but $\varphi(\bar{x}) \leq \lambda$ for any $x' \in I$, the conclusion is immediate, likewise in Theorem 4.2. □

Remark 4.3 The preceding theorem cannot be inverted. If we have $\varphi(x') < \limsup_{\|y'\| \to +\infty} \varphi(y')\ \ \forall x' \in I$, it is possible that $O^{E_+}(A) \neq \emptyset$. In fact, let $n = 3$ and $I := cl(E_+) = \{(x, y) \in \mathbb{R}^2 :\ x \geq 0,\ y \geq 0\}$. When $(x, y) \in I$, let $\varphi(x, y) := (1 - y) \arctan x$ and $A := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in I, z \leq \varphi(x, y)\}$.

We obtain $\limsup_{\|(x,y)\| \to +\infty} \varphi(x, y) = \pi/2$ and $\varphi(x, y) < \pi/2\ \ \forall\,(x, y) \in I$, but $O^{E_+}(A) \neq \emptyset$ (for example $(1, 1, 0) \in O^{E_+}(A)$, as one can easily verify).

Remark 4.4 a) The previous existence results, obtained when the improvement set is E_+, may be regarded as an extension of Theorem 3.2 in [14] in the finite dimensional setting.

In fact, let $A = \{(x, y) \in \mathbb{R}^2 : y \leq 1 + x^2 e^{-|x|}\}$, our Theorem 4.2 and Proposition 3.5 allow us to conclude that $O^{E_+}(A)$ is not empty and closed. However, Theorem 3.2 with condition (C) or its Corollary 3.1 in [14] is not applicable. In fact, condition (C) requires the existence of a nonempty compact set $D \subseteq A$ such that any point outside D cannot be weak Pareto efficient. Easily, we can compute that $O^{E_+}(A) = \left\{(-2,\ 1 + \frac{4}{e^2})\right\} \cup \{(x_o,\ 1 + x_o^2\, e^{-x_o}),\quad x_o \geq 2\}$.

b) First of all, we observe that, in our setting, Problem (3.1) in [15] becomes

$$\text{find } \bar{\mathbf{x}} \in A : \bar{\mathbf{x}} + \mathbf{e} \notin A \quad \text{for all} \quad \mathbf{e} \in E_+ = int(P)$$

with $P = \mathbb{R}^n_+$ and $A \subset \mathbb{R}^n$ a closed set. If we consider A, defined in the part a) of this remark, we see that Theorem 3.5 of [15] is not applicable. In fact, A is only closed and not convex, but especially condition (*) is not verified.

For the reader's convenience, we recall such condition in our setting:

(*) for any sequence $\{\mathbf{x}_n\}$ in A satisfying:

(i) $\|\mathbf{x}_n\| \to +\infty$, $\dfrac{\mathbf{x}_n}{\|\mathbf{x}_n\|} \to \mathbf{w}$ for some $\mathbf{w} \in R_1$, where $R_1 := \bigcap_{\mathbf{y} \in A}\{\mathbf{w} \in A^\infty : \lambda\mathbf{w} \notin -E_+\ \forall\,\lambda > 0\}$ and A^∞ is the recession cone of A, i.e., $A^\infty = \{\mathbf{x} \in \mathbb{R}^n : \exists\, t_k \downarrow 0,\ \exists\, \mathbf{x}_k \in A,\ t_k\mathbf{x}_k \to \mathbf{x}\}$, and

(ii) $\forall\, \mathbf{y} \in A\ \exists\, n_y$ such that $\mathbf{y} \notin \mathbf{x}_n + E_+\ \forall\, n > n_y$, we assume the existence of $\mathbf{u} \in A$ and $\bar{n}$, such that $\|\mathbf{u}\| < \|\mathbf{x}_{\bar{n}}\|$ and $\mathbf{u} = \mathbf{x}_{\bar{n}} + \mathbf{e}$ for some $\mathbf{e} \in \mathbb{R}^n_+$.
Then, if we consider the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$, defined by $\mathbf{x}_n = (n, 1 + n^2 e^{-n})$, $n \geq 2$, we may verify that $\lim_n \dfrac{\mathbf{x}_n}{\|\mathbf{x}_n\|} = (1, 0) \in R_1,\ A \cap (\mathbf{x}_n + E_+) = \emptyset\ \forall\, n \geq 2$, but, for every $\mathbf{u} \in A$ and $n \geq 2$ such that $\|\mathbf{u}\| < \|\mathbf{x}_n\|$, we have $\mathbf{u} \notin \mathbf{x}_n + \mathbb{R}^n_+$.

Now we give a result when $A \subset \mathbb{R}^n$ is closed, $n \geq 1$, and the improvement set is always E_+. To our knowledge, there are no existence results about weak Pareto efficiency involving the distance between a set and the cone $\mathbb{R}^n_{++}$.

Theorem 4.4 *Let $A \subset \mathbb{R}^n,\ n \geq 1$, a closed and nonempty set such that*

$$d(A, E_+) < \liminf_{r\to+\infty} d(A \cap \mathcal{C}B(\mathbf{0}, r), E_+ \cap \mathcal{C}B(\mathbf{0}, r)), \tag{4}$$

where $\mathcal{C}B(\mathbf{0}, r)$ denotes the complement of the ball with center the origin and radius $r > 0$. Then $O^{E_+}(A) \neq \emptyset$.

Proof We first take into account the case $d(A, E_+) = 0$. Since $\mathbf{y} + E_+ \subset E_+$ for each $\mathbf{y} \in E_+$, thanks to Remark 2.1, it turns out

$$d(A \cap \mathcal{C}B(\mathbf{0}, r),\ E_+ \cap \mathcal{C}B(\mathbf{0}, r)) \leq d(A \cap \mathcal{C}B(\mathbf{0}, r),\ (\mathbf{y} + E_+) \cap \mathcal{C}B(\mathbf{0}, r))$$

Hence it is possible to find $\mathbf{y}_o \in E_+$ such that

$$0 < d(A, \mathbf{y}_o + E_+) < \liminf_{r\to+\infty} d(A \cap \mathcal{C}B(\mathbf{0}, r), (\mathbf{y}_o + E_+) \cap \mathcal{C}B(\mathbf{0}, r))$$

If now, for simplicity, we carry out a change of coordinates so that the new origin is at $\mathbf{y}_o$, from the previous inequalities it follows

$$0 < d(A, E_+) < \liminf_{r\to+\infty} d(A \cap \mathcal{C}B(-\mathbf{y}_o, r), E_+ \cap \mathcal{C}B(-\mathbf{y}_o, r))$$

Thus we are able to consider only the case when $d(A, E_+) > 0$. Taking into account the definition of distance between two sets, there exist two sequences $\{\mathbf{x}_m\}_{m\in\boldsymbol{N}} \subset A,\ \{\mathbf{y}_m\}_{m\in\boldsymbol{N}} \subset E_+$ such that

$$\lim_{m\to+\infty} d(\mathbf{x}_m, \mathbf{y}_m) = d(A, E_+) = \inf\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in A,\ \mathbf{y} \in E_+\}$$

By (4), at least one between $\{\mathbf{x}_m\}$ and $\{\mathbf{y}_m\}$ is bounded. Without loss of generality we suppose that such a sequence is $\{\mathbf{x}_m\}$. Always according to (4), we have $d(A, E_+) \in \mathbb{R}$, so that $\inf\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in A,\ \mathbf{y} \in E_+\} \in \mathbb{R}$ and consequently also $\{\mathbf{y}_m\}$ is bounded. Therefore, there exist two subsequences, which we still denote by $\{\mathbf{x}_m\}$ and $\{\mathbf{y}_m\}$ for convenience, such that $\lim_{m\to+\infty} \mathbf{x}_m = \hat{\mathbf{a}} \in A;\ \lim_{m\to+\infty} \mathbf{y}_m = \hat{\mathbf{e}} \in cl(E_+)$ and $d(\hat{\mathbf{a}}, \hat{\mathbf{e}}) = d(A, cl(E_+)) = d(A, E_+)$. We claim that $\hat{\mathbf{a}} \in O^{E_+}(A)$. Indeed, we know that $\hat{\mathbf{a}} \notin cl(E_+)$ since $d(A, E_+) > 0$, and thus at least one component of $\hat{\mathbf{a}}$ is negative;

suppose (for simplicity) such a component is $\hat{a}_1$. Then let $\mathbf{h} \in E_+$ and $\alpha \in \mathbb{R}$ such that

$$0 < \alpha < \min\{h_1, -\hat{a}_1\} \tag{5}$$

If $\mathbf{w} = (h_1 - \alpha, h_2, \ldots, h_n)$, from (5), we have $\mathbf{w} \in E_+$. We now evaluate $d(\hat{\mathbf{a}} + \mathbf{h}, \hat{\mathbf{e}} + \mathbf{w}) = \left\{(\hat{a}_1 + \alpha - \hat{e}_1)^2 + \sum_{i=2}^{n}(\hat{a}_i - \hat{e}_i)^2\right\}^{1/2}$. From (5) it turns out $\hat{a}_1 + \alpha < 0$ and $|\hat{a}_1 + \alpha - \hat{e}_1| = \hat{e}_1 - \hat{a}_1 - \alpha < \hat{e}_1 - \hat{a}_1$. So we can establish that $d(\hat{\mathbf{a}} + \mathbf{h}, \hat{\mathbf{e}} + \mathbf{w}) < d(\hat{\mathbf{a}} + \mathbf{h}, \hat{\mathbf{e}} + \mathbf{h}) = d(\hat{\mathbf{a}}, \hat{\mathbf{e}})$ and, since $\hat{\mathbf{e}} + \mathbf{w} \in E_+$, it follows that $d(\hat{\mathbf{a}} + \mathbf{h}, E_+) < d(A, E_+)$, which yields $\hat{\mathbf{a}} + \mathbf{h} \notin A$, that is $\hat{\mathbf{a}} \in O^{E_+}(A)$. □

Remark 4.5 (i) Condition (4) in Theorem 4.4 is only a sufficient condition to have $O^{E_+}(A) \neq \emptyset$. In fact, if $f(x) = \begin{cases} -x-2 & \text{for } x \in]-\infty, -1] \\ -1 & \text{for } x \in]-1, +\infty] \end{cases}$ and $A = \{(x, y) \in \mathbb{R}^2 : y \leq f(x)\}$, then $O^{E_+}(A) \neq \emptyset$ (as one can easily verify) and $\liminf_{r\to+\infty} d(A \cap \mathcal{C}B(\mathbf{0}, r), E_+ \cap \mathcal{C}B(\mathbf{0}, r)) = 1 = d(A, E_+)$.

(ii) If $d(A, E_+) = \liminf_{r\to+\infty} d(A \cap \mathcal{C}B(\mathbf{0}, r), E_+ \cap \mathcal{C}B(\mathbf{0}, r))$, as in the above example, $O^{E_+}(A)$ can be empty. Let $A = \{(x, y) \in \mathbb{R}^2 : y \leq \exp(x)\}$, it follows $d(A, E_+) = 0 = \liminf_{r\to+\infty} d(A \cap \mathcal{C}B(\mathbf{0}, r), E_+ \cap \mathcal{C}B(\mathbf{0}, r))$ and $O^{E_+}(A) = \emptyset$.

(iii) In the previous example $d(A, E_+) = 0$, but $O^{E_+}(A)$ may be empty also when $0 < d(A, E_+) = \liminf_{r\to+\infty} d(A \cap \mathcal{C}B(\mathbf{0}, r), E_+ \cap \mathcal{C}B(\mathbf{0}, r))$.
In fact, let $A = \left\{(x, y) \in \mathbb{R}^2 : -2 \leq x < -1,\ y \geq -\frac{1}{x+1}\right\}$. We have $\liminf_{r\to+\infty} d(A \cap \mathcal{C}B(\mathbf{0}, r), E_+ \cap \mathcal{C}B(\mathbf{0}, r)) = 1 = d(A, E_+)$ and $O^{E_+}(A) = \emptyset$.

Theorem 4.5 *Let $A \subset \mathbb{R}^n,\ n \geq 1$, be a closed and nonempty set such that*

$$\limsup_{r\to+\infty} \left\{\max_{1\leq i\leq n} x_i : \mathbf{x} \in A \cap \mathcal{C}B(\mathbf{0}, r)\right\} = +\infty \tag{6}$$

Then there exist $\lambda_i \in \mathbb{R} \cup \{+\infty\}$ such that if $\mathbf{x} \in A$ satisfies $x_i < \lambda_i$ for every $i \in \{1, \ldots, n\}$, it turns out $\mathbf{x} \notin O^{E_+}(A)$.

Proof Note that assumption (6) implies the existence of $r_o > 0$ such that for each $r \geq r_o$ it turns out $A \cap \mathcal{C}B(\mathbf{0}, r) \neq \emptyset$. Thus let

$$\lambda_1 := \limsup_{r\to+\infty} \{x_1 : \mathbf{x} \in A \cap \mathcal{C}B(\mathbf{0}, r)\} \in \mathbb{R} \cup \{+\infty\}$$

and T_1 the set of the sequences $\{\mathbf{x}_k^{(1)}\}_{k\in\mathbb{N}} \subset A$ such that $\lim\limits_{k\to+\infty} \|\mathbf{x}_k^{(1)}\| = +\infty$ and $\lim\limits_{k\to+\infty} (x_k^{(1)})_1 = \lambda_1$. (We note that T_1 is not empty by definition of lim sup). Let $\{\mathbf{x}_k^{(1)}\}_{k\in\mathbb{N}} \in T_1$. We define $\lambda_2(\mathbf{x}_k^{(1)}) := \limsup_{k\to+\infty} (x_k^{(1)})_2 \in \mathbb{R} \cup \{+\infty\}$ and $\lambda_2 := \sup\{\lambda_2(\mathbf{x}_k^{(1)}) : \{\mathbf{x}_k^{(1)}\}_{k\in\mathbb{N}} \in T_1\}$. If $\lambda_2 \in \mathbb{R}$, then for every $\varepsilon > 0$, there exist $\{\mathbf{x}_k^{(1)}\}_{k\in\mathbb{N}} \in T_1$ and $k_o \in \mathbb{N}$ such that $\lambda_2 - \varepsilon < (x_k^{(1)})_2 \le \lambda_2 + \varepsilon$ for infinitely many $k \ge k_o$. If $\lambda_2 = +\infty$, then, for every $\varepsilon > 0$, there exist $\{\mathbf{x}_k^{(1)}\}_{k\in\mathbb{N}} \in T_1$ and $k_o \in \mathbb{N}$ such that $(x_k^{(1)})_2 \ge \varepsilon$ for infinitely many $k \ge k_o$. Therefore, in both cases, there exists a subsequence of $\{\mathbf{x}_k^{(1)}\}_{k\in\mathbb{N}}$, denoted by $\{\mathbf{x}_k^{(2)}\}_{k\in\mathbb{N}}$, such that $\lim\limits_{k\to+\infty} \|\mathbf{x}_k^{(2)}\| = +\infty$, $\lim\limits_{k\to+\infty} (x_k^{(2)})_1 = \lambda_1$, $\lim\limits_{k\to+\infty} (x_k^{(2)})_2 = \lambda_2$. Let T_2 be the set of the subsequences $\{\mathbf{x}_k^{(2)}\}_{k\in\mathbb{N}}$, which satisfy the above properties. At the n-th step, we determine a subsequence of $\{\mathbf{x}_k^{(n-1)}\}_{k\in\boldsymbol{N}}$, denoted by $\{\mathbf{x}_k^{(n)}\}_{k\in\mathbb{N}}$, $\{\mathbf{x}_k^{(n)}\}_{k\in\mathbb{N}} \subset A$, such that $\lim\limits_{k\to+\infty} \|\mathbf{x}^{(n,k)}\| = +\infty$ and we have $\lim\limits_{k\to+\infty} (x_k^{(n)})_i = \lambda_i \ \ \forall i = 1, 2, \dots, n$, where $\lambda_i(\{\mathbf{x}_k^{(i-1)}\}_{k\in\mathbb{N}}) = \limsup_{k\to+\infty} (x_k^{(i-1)})_i$ and $\lambda_i = \sup\{\lambda_i(\{x_k^{(i-1)}\}_{k\in\mathbb{N}}) : \{\mathbf{x}_k^{(i-1)}\}_{k\in\mathbb{N}} \in T_{i-1}\}$.

Consider now $\mathbf{x} \in A$ and suppose

$$x_i < \lambda_i \quad (i = 1, 2, \dots, n) \tag{7}$$

where the λ_is are constructed as before (of course, if for some i we have $\lambda_i = +\infty$, inequality (7) is automatically verified).

From the above construction, there exists a sequence $\{\mathbf{y}_k\}_{k\in\mathbb{N}} \subset A$ such that

$$\lim_k (y_k)_i = \lambda_i \quad (i = 1, 2, \dots, n) \tag{8}$$

From (7) and (8), we obtain the existence of $k_o \in \mathbb{N}$ such that $x_i < (y_{k_o})_i$ $(i = 1, 2, \dots, n)$. The last inequality means $\mathbf{y}_{k_o} - \mathbf{x} \in E_+$, therefore, $\mathbf{x} \notin O^{E_+}(A)$. $\square$

Remark 4.6 (i) If $A \subset \mathbb{R}^n$, $n \ge 1$, is a closed and nonempty set and there exist $\hat{i} \in \{1, 2, \dots, n\}$, $\bar{r} > 0$, $\hat{\mathbf{x}} \in A \cap \mathcal{C}B(\mathbf{0}, \bar{r})$ such that $\hat{x}_{\hat{i}} \ge x_{\hat{i}}$ for every $\mathbf{x} \in A \cap \mathcal{C}B(\mathbf{0}, \bar{r})$, then $O^{E_+}(A) \ne \emptyset$.
First, we observe that it is not restrictive to choose the coordinate system in such a way that $\hat{x}_i \ge 0$ for every $i = 1, 2, \dots, n$. If $\mathbf{e} \in E_+$, it is easy to verify that $\hat{\mathbf{x}} + \mathbf{e} \in A \cap \mathcal{C}B(\mathbf{0}, \bar{r})$, therefore $\hat{\mathbf{x}} + \mathbf{e} \notin A$ and finally $\hat{\mathbf{x}} \in O^{E_+}(A)$.

(ii) If, for every $\varepsilon > 0$, there exists $\hat{\mathbf{x}} \in A$ such that $\hat{x}_i \ge \varepsilon$ $(i = 1, 2, \dots, n)$, where A is always a closed and nonempty set of $\mathbb{R}^n$, then $O^{E_+}(A) = \emptyset$. The simple proof of this assertion can be left to the reader.

Next theorem gives a necessary condition in order that $O^E(A) \ne \emptyset$, when $E \in \Im^n$, $E \ne E_+$, satisfies a particular property.

Theorem 4.6 *Let $A \subset \mathbb{R}^n$ $(n \ge 2)$ be a nonempty set and $E \in \Im^n$ such that $E \supset \{\mathbf{x} \in \mathbb{R}^n : x_i \le 0$ and $x_n > -\alpha x_i$ $(i = 1, \dots, n-1)\}$ (where $\alpha > 0$).*

Then a necessary condition for $O^E(A) \neq \emptyset$ *is that there exists* $\mathbf{x}' \in A$ *such that, for every* $\mathbf{x}'' \in A$*, for which* $x_i' - x_i'' > 0$ *for each* $i = 1, \ldots, n-1$*, we have* $\dfrac{x_n'' - x_n'}{x_j' - x_j''} \leq \alpha$ for at least a value of $j \in \{1, 2, \ldots, n-1\}$.

Proof Suppose, on the contrary, that for every $\mathbf{x}' \in A$, there exists $\mathbf{x}'' \in A$ with $x_i' - x_i'' > 0 \ \forall i \in \{1, \ldots, n-1\}$ such that $\dfrac{x_n'' - x_n'}{x_i' - x_i''} > \alpha$ for each $i \in \{1, \ldots, n-1\}$.

This inequality gives $x_n'' - x_n' > -\alpha\,(x_i'' - x_i')$ for every $i = 1, \ldots, n-1$, so that, if we consider $\mathbf{e} = \mathbf{x}'' - \mathbf{x}'$, it turns out $0 < -\alpha\, e_i < e_n \ \ (i = 1, \ldots, n-1)$, hence $\mathbf{e} \in E$. Since $\mathbf{x}' + \mathbf{e} = \mathbf{x}'' \in A$, we have $\mathbf{x}' \notin O^E(A)$ and, in virtue of arbitrariness of $\mathbf{x}'$, it follows $O^E(A) = \emptyset$. □

5 A Situation in the Case $n = 1$.

In the interesting paper [12], under the convergence of a sequence of sets in the sense of Wijsman, Ke Quan Zhao and Xin Min Yang prove a stability result with perturbations concerning improvement sets. We refer two results of this paper for the reader's convenience.

Theorem 5.1 ([12], Theorem 3.1) *Let* $E,\ E_m \in \Im^n$ *and* $A,\ E$ *be closed. Suppose that* $A_m \to A$ *and* $E_m \to E$ *in the sense of Wijsman.*

If $x_m \in O^{int(E_m)}(A_m)$ *and* $x_m \to x_o$ $(m \to +\infty)$*, then* $x_o \in O^{int(E)}(A)$.

Theorem 5.2 ([12], Theorem 3.2) *Let* $E,\ E_m \in \Im^n$ *and* $A,\ E$ *be closed. Suppose that* $A_m \to A$ *and* $E_m \to E$ *in the sense of Wijsman. Then, for every* $x \in \mathbb{R}^n$, $\liminf_m\ d(x, O^{int(E_m)}(A_m)) \geq d(x, O^{int(E)}(A))$.

Concerning these results, the authors Zhao and Yang proposed the question if, in the hypothesis, the sets $int(E_m)$ and $int(E)$ could be replaced by E_m and E, respectively. The following examples show that the answer is negative.

Example 5.1 Let $A_m = \left\{0,\ -1 + \dfrac{1}{m}\right\}$ $(m \in \mathbb{N},\ m \geq 2)$, $A = \{0,\ -1\}$ be subsets of $\mathbb{R}$, $E_m = E = \{x \in \mathbb{R} : x \geq 1\}$ and $x_m = -1 + \dfrac{1}{m}$.

Then it is easy to see that $x_m \in O^{E_m}(A_m)$, $\lim_m x_m = -1$, but $-1 \notin O^E(A)$.

Example 5.2 Let again $A_m = \left\{0,\ -1 + \dfrac{1}{m}\right\}$ $(m \geq 2)$, $A = \{0,\ -1\}$, $E_m = E = \{x \in \mathbb{R} : x \geq 1\}$. Then it follows $O^{E_m}(A_m) = \left\{0,\ -1 + \dfrac{1}{m}\right\}$, $O^E(A) = \{0\}$ and, if $x = -1$, it turns out $\lim_m d(-1,\ O^{E_m}(A_m)) = \lim_m \dfrac{1}{m} = 0$, while $d(-1,\ O^E(A)) = 1$.

6 Perspectives

It would be interesting to obtain existence results via topological properties of the sets for the solutions of the problem formulated by Flores-Bazán and Hernández in [3] and [1] and briefly described in the Introduction. This improvement may be stimulating in both finite and infinite dimensions, also for the applications in mathematical economics. The importance of finding existence results also through a topological process in addition to via scalarization and via subdifferentials (for these two last methods see [3] and [1]) lies in the fact that most numerical procedures or location methods such as the iterative and heuristic ones yield feasible points near the exact solution.

A topic of future research will be also the study of the existence for approximate solutions of problems associated with that quoted above and introduced in [3].

7 Conclusions

In this paper, we are mainly interested in the existence of optimal points for multicriteria situations, keeping into account special sets, called improvement sets, as described in the Introduction. In a finite dimensional setting, we have considered especially the case in which the order set is the interior of the nonnegative orthant and also the case where it is a generic improvement set. In both cases, we have provided some conditions (necessary, sufficient or both) for the existence of optimal points, illustrating their novelty. In particular, our Theorem 4.2 and Theorem 4.3 improve the existence results given in [14] and [15].

Acknowledgments The authors are grateful to the anonymous Referees for their helpful suggestions and remarks, which helped to improve the original version of the paper.

References

1. Flores-Bazán, F., Hernández, E.: Optimality conditions for a unified vector optimization problem with not necessarily preordering relations. J. Glob. Optim. **56**(2), 299–315 (2013)
2. Makarov, V.L., Levin, M.J., Rubinov, A.M.: Mathematical economic theory: pure and mixed types of economic mechanisms. North-Holland Publishing Co., Amsterdam (1995)
3. Flores-Bazán, F., Hernández, E.: A unified vector optimization problem: complete scalarizations and applications. Optimization **60**(12), 1399–1419 (2011)
4. Ha, T.X.D.: Optimality conditions for several types of efficient solutions of set-valued optimization problems. In: Pardalos, P., Rassias, ThM, Khan, A.A. (eds.) Nonlinear Analysis and Variational Problems, pp. 305–324. Springer, New York (2010)
5. Chicco, M., Mignanego, F., Pusillo, L., Tijs, S.: Vector optimization problems via improvement sets. J. Optim. Theory Appl. **150**(3), 516–529 (2011)
6. Gutiérrez, C., Jiménez, B., Novo, V.: Improvement sets and vector optimization. Eur. J. Oper. Res. **223**(2), 304–311 (2012)
7. Gutiérrez, C., Jiménez, B., Novo, V.: A unified approach and optimality conditions for approximate solutions of vector optimization problems. SIAM J. Optim. **17**(3), 688–710 (2006)
8. Gutiérrez, C., Jiménez, B., Novo, V.: Optimality conditions via scalarization for a new ε-efficiency concept in vector optimization problems. Eur. J. Oper. Res. **201**(1), 11–22 (2010)
9. Oppezzi, P., Rossi, A.M.: Existence and convergence of Pareto minima. J. Optim. Theory Appl. **128**(3), 653–664 (2006)

10. Flores-Bazán, F., Hernández, E., Novo, V.: Characterizing efficiency without linear structure: a unified approach. J. Glob. Optim. **41**(1), 43–60 (2008)
11. Pusillo, L., Tijs, S.: E-Equilibria for multicriteria games. Advances in dynamic games, Ann. Internat. Soc. Dynam. Games, 12, pp. 217–228. Birkhuser-Springer, New York (2012)
12. Zhao, K.Q., Yang, X.M.: A unified stability result with perturbations in vector optimization. Optim. Lett. **7**(8), 1913–1919 (2013)
13. Xanthos, F.: Non-existence of weakly Pareto optimal allocations. Econ. Theory Bull. **2**, 137–146 (2014)
14. Flores-Bazán, F., Vera, C.: Characterization of the nonemptiness and compactness of solution sets in convex and nonconvex vector optimization. J. Optim. Theory Appl. **130**(2), 185–207 (2006)
15. Flores-Bazán, F.: Ideal, weakly efficient solutions for vector optimization problems. Math. Program. Ser. A **93**(3), 453–475 (2002)