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Dirichlet Problem for a Divergence Form Elliptic Equation with Unbounded Coefficients in an Unbounded Domain (*).

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Abstract. – *We prove existence and uniqueness of the solution of the Dirichlet problem for a class of elliptic equations in divergence form with discontinuous and unbounded coefficients in unbounded domains.*

1. – Introduction.

In 1985 in two interesting papers [4], [5] P. L. Lions considered the Dirichlet problem

$$\begin{cases} a_0(u, v) = \langle T, v\rangle \qquad \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega) \end{cases} \tag{1}$$

where T is given in $H^{-1}(\Omega)$. The bilinear form $a_0(\cdot, \cdot)$ is defined as follows:

$$a_0(u, v) := \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b_i u_{x_i} v + cuv \right\} dx \tag{2}$$

where $a_{ij} \in L^\infty(\Omega)$, $\sum_{i,j=1}^n a_{ij} t_i t_j \geq \nu |t|^2$ for all $t \in \mathbb{R}^n$ (with ν a positive costant), $b_i \in L^\infty(\Omega)$ $(i = 1, 2, \ldots, n)$, $c \geq c_0$ (positive constant). The open set Ω, contained in $\mathbb{R}^n$, is not supposed to be bounded. The main result of the works by P. L. Lions is that, under the hypotheses above, there exists a unique solution of problem (1) and the a priori inequality

$$\|u\|_{H^1(\Omega)} \leq K_1 \|T\|_{H^{-1}(\Omega)} \tag{3}$$

holds, where K_1 is a constant depending on n and the coefficients of the bilinear form $a_0(\cdot, \cdot)$.

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The aim of the present note is to extend these results assuming the coefficients b_i to belong only to the space $X^p(\Omega)$ $(i = 1, 2, \ldots, n)$ with $p > n$ (see Definition 1 below or [2]). The proof is similar to the one in [4], [5]; we add some new remarks (e.g. Lemma 3).

2. - Preliminaries.

Let Ω be an open subset of $\mathbb{R}^n$; for simplicity we assume $n \geq 3$.

DEFINITION 1. - *Let*

$$\omega(f, p, \delta) := \sup\{\|f\|_{L^p(E)}: E \text{ measurable}, \ \ E \subset \Omega, \ \ \text{meas } E \leq \delta\}$$

$$X^p(\Omega) := \{f \in L^p_{\text{loc}}(\Omega): \omega(f, p, \delta) < +\infty \ \forall \delta > 0\}$$

$$X^p_0(\Omega) := \{f \in X^p(\Omega): \lim_{\delta \to 0+} \omega(f, p, \delta) = 0\}$$

For further properties of these spaces, see [2].

LEMMA 1 (Uniqueness). - *If* $a_{ij} \in L^\infty(\Omega)$ $(i, j = 1, 2, \ldots n)$, $\sum_{i,j=1}^n a_{ij} t_i t_j \geq \nu |t|^2$ *for all* $t \in \mathbb{R}^n$, $b_i \in X^n_0(\Omega)$ $(i = 1, 2, \ldots, n)$, $c \geq c_0$ *in* Ω *(*ν, c_0 *positive constants),* $c \in X^{n/2}_0(\Omega)$, *then problem* (1) *(with the bilinear form* $a_0(\cdot, \cdot)$ *defined in* (2)) *has at most one solution.*

PROOF. - It is sufficient to show that if $u \in H^1_0(\Omega)$, $a_0(u, v) \leq 0$ $\forall v \in H^1_0(\Omega)$, $v \geq 0$ in Ω, then $u \leq 0$ a.e. in Ω. Arguing by contradiction, suppose that $m := \operatorname{ess\,sup}_\Omega u > 0$. Choose t with $0 < t < m$ and let $u_t := \max(u - t, 0)$. Since $u \in H^1_0(\Omega)$, in particular $u \in L^2(\Omega)$, then $u_t > 0$ only in a set of finite measure. Therefore, replacing v with u_t in (1) and observing that $u_{x_i} = (u_t)_{x_i}$ a.e. in $\Omega_t := \{x \in \Omega: t < u(x) < m\}$, it follows from the assumptions above that

$$c_0 \|u_t\|^2_{L^2(\Omega)} + \nu \|(u_t)_x\|^2_{L^2(\Omega)} \leq S \sum_{i=1}^n \|b_i\|_{L^n(\Omega_t)} \|(u_t)_x\|^2_{L^2(\Omega_t)}, \tag{4}$$

where S denotes the constant in the Sobolev inequality

$$\|\phi\|_{L^{2n/(n-2)}(\mathbb{R}^n)} \leq S \|\phi_x\|_{L^2(\mathbb{R}^n)} \qquad \forall \phi \in C^1_0(\mathbb{R}^n)$$

(It is well known that the constant S depends only on n: see e.g. [8].) We can choose t so close to m that meas Ω_t be as small as we like, and since $b_i \in X^n_0(\Omega)$, we obtain $\sum_{i=1}^n \|b_i\|_{L^n(\Omega_t)} < \nu/S$. Then from (4) we get

$$u_t = 0 \quad \text{ a.e. in } \Omega,$$

which is a contradiction, since $m := \operatorname{ess\,sup}_\Omega u > t$. ■

The following lemma is a more precise version of the classical Sobolev inequality.

LEMMA 2. – *Let Q be a cube in $\mathbb{R}^n$ with side length r, and $u \in H^1(Q)$. Then there exists a constant K_2, depending only on n, such that*

$$\|u\|_{L^{2n/(n-2)}(Q)} \leq K_2[(1/r)\|u\|_{L^2(Q)} + \|u_x\|_{L^2(Q)}]. \tag{5}$$

PROOF. – A proof of this result can be found e.g. in [3]; we give an outline only for convenience of the reader. First of all, it is sufficient to consider the case $r = 1$, and the general case easily follows by a change of variables (dilation).

We can use inequalities (5.7), (5.8) of [3] replacing Ω with Q, $l = 1$, $p = 2$, and obtain

$$\|u\|_{L^{2n/(n-2)}(Q)} \leq 2^{(n-2)/(2n-2)}\|u\|_{L^{(2n-2)/(n-2)}(Q)} + 4(n-1)/(n-2)\sum_{i=1}^{n}\|u_{x_i}\|_{L^2(Q)}. \tag{6}$$

Since $2 < 2(n-1)/(n-2) < 2n/(n-2)$, then from Lemma 3.1 of [3], with $p_1 = 2$, $p = 2(n-1)/(n-2)$, $p_2 = 2n/(n-2)$, $\varepsilon = 2^{-(3n-4)/(2n-2)}$, we get:

$$\|u\|_{L^{(2n-2)/(n-2)}(Q)} \leq 2^{-(3n-4)/(2n-2)}\|u\|_{L^{2n/(n-2)}(Q)} + 2^{n(3n-4)/(2n-2)(n-2)}\|u\|_{L^2(Q)}. \tag{7}$$

We combine (7) and (6) and finally get

$$\|u\|_{L^{2n/(n-2)}(Q)} \leq 2^{(3n-4)/(n-2)}\|u\|_{L^2(Q)} + 8(n-1)/(n-2)\sum_{i=1}^{n}\|u_{x_i}\|_{L^2(Q)}$$

whence the conclusion (5) easily follows. ■

DEFINITION 2. – (Stampacchia [7]). *The bilinear form*

$$a(u, v) := \int_\Omega \left\{\sum_{i,j=1}^{n} a_{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} v + \sum_{i=1}^{n} d_i u v_{x_i} + cuv\right\} dx \tag{8}$$

is said to be coercitive on $H_0^1(\Omega)$ if there exists a positive constant c_1 such that

$$a(u, u) \geq c_1\|u\|^2_{H_0^1(\Omega)} \qquad \forall u \in H_0^1(\Omega).$$

The following result is an extension of theorem 3.2 of Stampacchia [7].

LEMMA 3. – *Suppose a_{ij} $(i, j = 1, 2, \ldots, n)$ as in Lemma 1, $b_i, d_i \in X_0^n(\Omega)$ $(i = 1, 2, \ldots, n)$, $c \in X_0^{n/2}(\Omega)$, $a(\cdot, \cdot)$ defined as in* (8).

Then there exists a constant λ_0 (depending on the coefficients of $a(\cdot, \cdot)$) such that the bilinear form

$$a(u, v) + \lambda\int_\Omega uv\, dx$$

is coercitive on $H_0^1(\Omega)$ whenever $\lambda \geq \lambda_0$.

PROOF. – Let $\{Q_h\}_{h\in\mathbb{N}}$ be a family of open cubes in $\mathbb{R}^n$, with constant side length r, such that $\bigcup_{h=1}^{+\infty}\overline{Q_h} = \mathbb{R}^n$ and $Q_k \cap Q_h = \emptyset$ if $h \neq k$. By the assumptions above and Definition

1, we can choose $r>0$ such that

$$\sum_{i=1}^{n}\|b_i\|_{L^n(Q_h)}\leq \nu/8K_2,\qquad \sum_{i=1}^{n}\|d_i\|_{L^n(Q_h)}\leq \nu/8K_2,\qquad \|c\|_{L^{n/2}(Q_h)}\leq \nu/8K_2^2, \tag{9}$$

$$(h=1,2,\ldots).$$

Then, taking Lemma 2 into account, if $u\in H_0^1(\mathbb{R}^n)$ it turns out:

$$\sum_{i=1}^{n}\int_{Q_h}|b_i u_{x_i} u|\,dx\leq(\nu/8K_2)\|u\|_{L^{2n/(n-2)}(Q_h)}\|u_x\|_{L^2(Q_h)}\leq \tag{10}$$

$$\leq(\nu/8)\|u_x\|_{L^2(Q_h)}[(1/r)\|u\|_{L^2(Q_h)}+\|u_x\|_{L^2(Q_h)}]\leq$$

$$\leq(\nu/4)\|u_x\|^2_{L^2(Q_h)}+(\nu/32r^2)\|u\|^2_{L^2(Q_h)}\qquad(h=1,2,\ldots)$$

and, by the same procedure,

$$\sum_{i=1}^{n}\int_{Q_h}|d_i u_{x_i} u|\,dx\leq(\nu/4)\|u_x\|^2_{L^2(Q_h)}+(\nu/32r^2)\|u\|^2_{L^2(Q_h)}\qquad(h=1,2,\ldots) \tag{11}$$

$$\int_{Q_h}|cu^2|\,dx\leq\|c\|_{L^{n/2}(Q_h)}\|u\|^2_{L^{2n/(n-2)}(Q_h)}\leq \tag{12}$$

$$\leq(\nu/4)\|u_x\|^2_{L^2(Q_h)}+(\nu/4r^2)\|u\|^2_{L^2(Q_h)}\qquad(h=1,2,\ldots).$$

Now suppose $u\in H_0^1(\Omega)$; from (10) we easily deduce

$$\left|\int_{\Omega}\sum_{i=1}^{n}b_i u_{x_i} u\,dx\right|\leq\sum_{h=1}^{+\infty}\int_{\Omega\cap Q_h}\left|\sum_{i=1}^{n}b_i u_{x_i} u\right|dx\leq \tag{13}$$

$$\leq(\nu/4)\sum_{h=1}^{+\infty}\int_{\Omega\cap Q_h}u_x^2\,dx+(\nu/32r^2)\sum_{h=1}^{+\infty}\int_{\Omega\cap Q_h}u^2\,dx=(\nu/4)\|u_x\|^2_{L^2(\Omega)}+(\nu/32r^2)\|u\|^2_{L^2(\Omega)}$$

and similarly

$$\left|\int_{\Omega}\sum_{i=1}^{n}d_i u_{x_i} u\,dx\right|\leq\ldots\leq(\nu/4)\|u_x\|^2_{L^2(\Omega)}+(\nu/32r^2)\|u\|^2_{L^2(\Omega)}, \tag{14}$$

$$\left|\int_{\Omega}cu^2\,dx\right|\leq\ldots\leq(\nu/4)\|u_x\|^2_{L^2(\Omega)}+(\nu/4r^2)\|u\|^2_{L^2(\Omega)}. \tag{15}$$

From (13), (14), (15) and uniform ellipticity the conclusion follows, with $\lambda_0=5\nu/16r^2+\nu/4$ and $c_1=\nu/4$. ■

Following Stampacchia [7] we have, first of all, that the Dirichlet problem

$$\begin{cases} a(u, v) + \lambda \int_\Omega uv\,dx = \langle T, v\rangle \ \ \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega) \end{cases} \tag{16}$$

(with T given in $H^{-1}(\Omega)$) has a unique solution if $\lambda \geq \lambda_0$. Notice, furthermore, that the Dirichlet problem

$$\begin{cases} a(u, v) = \langle T, v\rangle \ \ \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega) \end{cases} \tag{17}$$

has a unique solution if the same holds in the particular case $\langle T, v\rangle = \int_\Omega wv\,dx$ with $w \in H_0^1(\Omega)$. In fact we have the following result:

LEMMA 4. – *Suppose that the Dirichlet problem*

$$\begin{cases} a(u, v) = \int_\Omega wv\,dx \ \ \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega) \end{cases} \tag{18}$$

has a unique solution whenever w is given in $H_0^1(\Omega)$, and the a priori inequality

$$\|u\|_{L^2(\Omega)} \leq K_3 \|w\|_{L^2(\Omega)}$$

holds. Then problem (17) *also has a unique solution, and it turns out*

$$\|u\|_{L^2(\Omega)} \leq K_4 \|T\|_{H^{-1}(\Omega)} \tag{19}$$

where K_4 depends on the coefficients of $a(\cdot, \cdot)$ (K_4 can be explicitly evaluated).

PROOF. – Let $\lambda \geq \lambda_0$ (defined in Lemma 3). According to what we observed before, the problem

$$\begin{cases} a(u_1, v) + \lambda \int_\Omega u_1 v\,dx = \langle T, v\rangle \ \ \forall v \in H_0^1(\Omega), \\ u_1 \in H_0^1(\Omega) \end{cases} \tag{20}$$

has a unique solution u_1, which satisfies the a priori inequality

$$\|u_1\|_{H^1(\Omega)} \leq (1/c_1) \|T\|_{H^{-1}(\Omega)} \tag{21}$$

where c_1 is the constant in Definition 2. (Inequality (19) can be easily proved by using the fact that the bilinear form $a(u, v) + \lambda \int_\Omega uv\,dx$ is coercive on $H_0^1(\Omega)$). Then we consider the problem

$$(22)\qquad \begin{cases} a(u_2, v) = \int_\Omega u_1 v\,dx \ \forall v \in H_0^1(\Omega), \\ u_2 \in H_0^1(\Omega) \end{cases}$$

which by hypothesis has a unique solution u_2, and the inequality

$$(23)\qquad \|u_2\|_{L^2(\Omega)} \leq K_3 \|u_1\|_{L^2(\Omega)}$$

holds. From (20), (22) we get

$$(24)\qquad \begin{cases} a(u_1 + \lambda u_2, v) = \langle T, v\rangle \ \forall v \in H_0^1(\Omega), \\ u_1 + \lambda u_2 \in H_0^1(\Omega) \end{cases}$$

i.e. $u_1 + \lambda u_2$ is a solution of problem (17), and it is unique by hypothesis. Furthermore from (21), (23) we deduce

$$(25)\qquad \|u_1 + \lambda u_2\|_{L^2(\Omega)} \leq (1/c_1)(1 + \lambda K_3)\|T\|_{H^{-1}(\Omega)}$$

whence (19), with $K_4 = (1/c_1)(1 + \lambda_0 K_3)$, and λ_0 as in Lemma 3. ∎

The following Lemma is an extension of a result by Miranda ([6], Theorem 4.1).

LEMMA 5. – *Let $u \in H_0^1(\Omega)$ be a solution of the equation*

$$(26)\qquad a(u, v) = \int_\Omega fv\,dx \ \forall v \in H_0^1(\Omega)$$

with $f \in L^q(\Omega)$ $\forall q \geq q_0$ (q_0 constant, $q_0 \geq 2$), $b_i \in X_0^n(\Omega)$ ($i = 1, 2, \ldots, n$), $d_i \in X^p(\Omega)$ with $p > n$, $c = c' + c''$, $c' \geq c_0$ (c_0 positive constant), $c' \in X^{n/2}(\Omega)$, $c'' \in X^{np/(n+p)}(\Omega)$.

Then there exist $\varepsilon > 0$, $\bar q \geq q_0$, $K_5 > 0$ such that if $\omega(d_i, n, 1) < \varepsilon$ ($i = 1, 2, \ldots, n$), $\omega(c'', n/2, 1) < \varepsilon$, then

$$\|u\|_{L^{\bar q}(\Omega)} \leq K_5 \|f\|_{L^{\bar q}(\Omega)}$$

PROOF. – By Remark 3 of [2] applied to the coefficients d_i, c'', it turns out $d_i \in X_0^n(\Omega)$ ($i = 1, 2, \ldots, n$), $c'' \in X_0^{n/2}(\Omega)$, so we can apply the Theorem of [2] obtaining $u \in L^\infty(\Omega)$. Therefore if $\gamma \in \mathbb{R}$, $\gamma \geq 0$, then $v := |u|^{\gamma+1}\operatorname{sign}(u) \in H_0^1(\Omega)$. By choosing v as a test function we find (since $v_{x_i} = (\gamma + 1)|u|^\gamma u_{x_i}$ a.e. in Ω):

$$(27)\qquad \sum_{i,j=1}^n \int_\Omega a_{ij} u_{x_i} v_{x_j}\,dx \geq \nu(\gamma + 1)\int_\Omega |u|^\gamma u_x^2\,dx\,.$$

Furthermore, let $\{Q_h\}_{h \in \mathbb{N}}$ be a family of cubes of side length $r > 0$, as in Lemma 3. We have, by Hölder's inequality and Lemma 2,

$$(28)\qquad \left|\sum_{i=1}^{n}\int_{Q_h} b_i u_{x_i} v\,dx\right| \leq \sum_{i=1}^{n}\|b_i\|_{L^n(Q_h)}\||u|^{\gamma/2}u_x\|_{L^2(Q_h)}\||u|^{\gamma^*}\|_{L^{2n/(n-2)}(Q_h)} \leq$$

$$\leq K_2\sum_{i=1}^{n}\|b_i\|_{L^n(Q_h)}\||u|^{\gamma/2}u_x\|_{L^2(Q_h)}\left[\frac{1}{r}\||u|^{\gamma^*}\|_{L^2(Q_h)} + \gamma^*\||u|^{\gamma/2}u_x\|_{L^2(Q_h)}\right] \leq$$

$$\leq K_2\sum_{i=1}^{n}\|b_i\|_{L^n(Q_h)}[(\gamma^* + \mu/2)\||u|^{\gamma/2}u_x\|^2_{L^2(Q_h)} + 1/(2r^2\mu)\||u|^{\gamma^*}\|^2_{L^2(Q_h)}]$$

Here $\gamma^* := 1 + \gamma/2$, $r > 0$ is as in Lemma 2 and $\mu > 0$ arbitrary. Now let us choose r such that $0 < r \leq 1$ and

$$(29)\qquad K_2\sum_{i=1}^{n}\|b_i\|_{L^n(Q_h)} \leq \nu/4 \qquad (h = 1, 2, \ldots)$$

this is possible according to the assumptions on the coefficients b_i $(i = 1, 2, \ldots, n)$. Furthermore choose μ in (28) such that

$$(30)\qquad \mu = \nu/(2c_0r^2).$$

Therefore from (28), (29), (30) we deduce

$$(31)\qquad \left|\sum_{i=1}^{n}\int_{Q_h} b_i u_{x_i} v\,dx\right| \leq$$

$$\leq [\nu(\gamma^* + \nu/(4c_0r^2))/4]\||u|^{\gamma/2}u_x\|^2_{L^2(Q_h)} + (c_0/4)\||u|^{\gamma^*}\|^2_{L^2(Q_h)} \qquad (h = 1, 2, \ldots).$$

If we choose

$$(32)\qquad \gamma := \max(1, \nu/(2c_0r^2), q_0 - 2)$$

we get

$$\nu/(4c_0r^2) \leq \gamma/2$$

With all these choices (31) becomes

$$(33)\qquad \left|\sum_{i=1}^{n}\int_{Q_h} b_i u_{x_i} v\,dx\right| \leq$$

$$\leq [\nu(1+\gamma)/4]\||u|^{\gamma/2}u_x\|^2_{L^2(Q_h)} + (c_0/4)\||u|^{\gamma^*}\|^2_{L^2(Q_h)} \qquad (h = 1, 2, \ldots)$$

whence, by summing on h, we finally get

$$(34)\qquad \left|\sum_{i=1}^{n}\int_{\Omega} b_i u_{x_i} v\,dx\right| \leq [\nu(1+\gamma)/4]\||u|^{\gamma/2}u_x\|^2_{L^2(\Omega)} + (c_0/4)\||u|^{\gamma^*}\|^2_{L^2(\Omega)}.$$

By a similar procedure we can evaluate the other terms of the bilinear form $a(u, v)$. We have (again, by Hölder's inequality and Lemma 2):

$$(35)\quad (\gamma+1)^{-1}K_2^{-1}\left|\sum_{i=1}^{n}\int_{Q_h} d_i u v_{x_i}\,dx\right| \leqslant$$

$$\leqslant K_2^{-1}\sum_{i=1}^{n}\|d_i\|_{L^n(Q_h)}\||u|^{\gamma/2}u_x\|_{L^2(Q_h)}\||u|^{\gamma^*}\|_{L^{2n/(n-2)}(Q_h)} \leqslant$$

$$\leqslant \sum_{i=1}^{n}\|d_i\|_{L^n(Q_h)}\||u|^{\gamma/2}u_x\|_{L^2(Q_h)}\left[\frac{1}{r}\||u|^{\gamma^*}\|_{L^2(Q_h)}+\gamma^*\||u|^{\gamma/2}u_x\|_{L^2(Q_h)}\right] \leqslant$$

$$\leqslant \sum_{i=1}^{n}\|d_i\|_{L^n(Q_h)}[(1+\gamma^*)\||u|^{\gamma/2}u_x\|_{L^2(Q_h)}^2+(1/4r^2)\||u|^{\gamma^*}\|_{L^2(Q_h)}^2] \qquad (h=1,2,\ldots)$$

whence, by summing over h

$$(36)\quad (\gamma+1)^{-1}K_2^{-1}\left|\sum_{i=1}^{n}\int_{\Omega} d_i u v_{x_i}\,dx\right| \leqslant$$

$$\leqslant \left[\sup_h \sum_{i=1}^{n}\|d_i\|_{L^n(Q_h)}\right]\left[(1+\gamma^*)\||u|^{\gamma/2}u_x\|_{L^2(\Omega)}^2+(1/4r^2)\||u|^{\gamma^*}\|_{L^2(\Omega)}^2\right]$$

Similarly again

$$(37)\quad \left|\int_{Q_h} c'' uv\,dx\right| \leqslant \int_{Q_h}|c''||u|^{\gamma+2}dx \leqslant \|c''\|_{L^{n/2}(Q_h)}\||u|^{\gamma^*}\|_{L^{2n/(n-2)}(Q_h)}^2 \leqslant$$

$$\leqslant K_2^2\|c''\|_{L^{n/2}(Q_h)}[\gamma^*\||u|^{\gamma/2}u_x\|_{L^2(Q_h)}+(1/r)\||u|^{1+\gamma/2}\|_{L^2(Q_h)}]^2 \leqslant$$

$$\leqslant 2K_2^2\|c''\|_{L^{n/2}(Q_h)}[(\gamma^*)^2\||u|^{\gamma/2}u_x\|_{L^2(Q_h)}^2+(1/r^2)\||u|^{\gamma^*}\|_{L^2(Q_h)}^2]$$

$$(h=1,2,\ldots)$$

and by summing over h

$$(38)\quad \left|\int_{\Omega} c'' uv\,dx\right| \leqslant 2K_2^2\left[\sup_h\|c''\|_{L^{n/2}(Q_h)}\right]\left[(\gamma^*)^2\||u|^{\gamma/2}u_x\|_{L^2(\Omega)}^2+(1/r^2)\||u|^{\gamma^*}\|_{L^2(\Omega)}^2\right].$$

From (27), (34), (36), (38) we easily get the result. In fact if we choose ε such that

$$(39)\quad 0<\varepsilon\leqslant \min\{c_0r^2/[K_2(\gamma+1)],\ c_0r^2/(8K_2^2),\ \nu/[2K_2(\gamma+4)],\ \nu/[K_2^2(\gamma+2)^2]\}$$

since $0 < r \leqslant 1$, then from (36), (38)

$$(40)\qquad \left|\sum_{i=1}^{n}\int_{\Omega} d_i u v_{x_i}\,dx\right| \leqslant [\nu(\gamma+1)/4]\| |u|^{\gamma/2}u_x\|^2_{L^2(\Omega)} + (c_0/4)\| |u|^{\gamma^*}\|^2_{L^2(\Omega)}$$

$$(41)\qquad \left|\int_{\Omega} c'' uv\,dx\right| \leqslant [\nu(\gamma+1)/4]\| |u|^{\gamma/2}u_x\|^2_{L^2(\Omega)} + (c_0/4)\| |u|^{\gamma^*}\|^2_{L^2(\Omega)}.$$

From (27), (34), (40), (41), using (26) we deduce

$$(42)\qquad \|f\|_{L^{\gamma+2}(\Omega)}\|u\|^{\gamma+1}_{L^{\gamma+2}(\Omega)} \geqslant$$

$$\geqslant \int_{\Omega} fv\,dx = a(u,v) \geqslant \nu(\gamma+1)\| |u|^{\gamma/2}u_x\|^2_{L^2(\Omega)} + c_0\| |u|^{\gamma^*}\|^2_{L^2(\Omega)} +$$

$$-(\nu/4+\nu/4+\nu/4)(\gamma+1)\| |u|^{\gamma/2}u_x\|^2_{L^2(\Omega)} - (3c_0/4)\| |u|^{\gamma^*}\|^2_{L^2(\Omega)} \geqslant (c_0/4)\|u\|^{\gamma+2}_{L^{\gamma+2}(\Omega)}$$

whence finally

$$(43)\qquad \|u\|_{L^{\gamma+2}(\Omega)} \leqslant (4/c_0)\|f\|_{L^{\gamma+2}(\Omega)}.$$

The assertion is therefore proved with $\overline{q} := \gamma + 2$, $K_5 := 4/c_0$, γ given by (32), and ε defined as in (39). ■

It is now convenient to define the «dual bilinear form» with respect to $a(u, v)$ as follows:

$$(44)\qquad a'(u,v) := a(v,u) \qquad \forall u,\ v \in H_0^1(\Omega)$$

It is clear that, going from $a(u, v)$ to $a'(u, v)$, we interchange the coefficients b_i with the d_i's. Using the fact that $L^p(\Omega)$ and $L^q(\Omega)$ are dual spaces if $1/p + 1/q = 1$, it is easy to prove that Lemma 5 is equivalent to the following:

LEMMA 5′. – *Let $w \in H_0^1(\Omega)$ be a solution of the equation*

$$(45)\qquad a(w,v) = \int_{\Omega} gv\,dx \ \forall v \in H_0^1(\Omega)$$

with $g \in L^p(\Omega)$ $\forall p \in (1, p_0)$ (p_0 constant, $p_0 \in (1, 2]$), $d_i \in X_0^n(\Omega)$, $b_i \in X^q(\Omega)$ with $q > n$ $(i = 1, 2, \ldots, n)$, $c = c' + c''$, $c' \in X^{n/2}(\Omega)$, $c' \geqslant c_0$ (c_0 positive constant), $c'' \in X^{nq/(n+q)}(\Omega)$. Then there exist $\varepsilon > 0$, $\overline{p} \in (1, p_0]$, $K_6 > 0$ such that if $\omega(b_i, n, 1) < \varepsilon$ $(i = 1, 2, \ldots, n)$, $\omega(c'', n/2, 1) < \varepsilon$, then

$$(46)\qquad \|w\|_{L^{\overline{p}}(\Omega)} \leqslant K_6\|g\|_{L^{\overline{p}}(\Omega)}$$

PROOF. – As in [4], [5], we may assume without loss of generality that Ω is bounded, provided the costants in the a priori inequalities we prove are independent on Ω. Notice also that we have supposed the coefficients b_i $(i = 1, 2, \ldots, n)$ to be sufficiently small, instead of the d_i's as in Lemma 5. Therefore it is possible to apply Lemma 5 provided we replace the bilinear form $a(u, v)$ with $a'(u, v) := a(v, u)$ since, as we have already remarked, in this way the roles of the coefficients b_i and d_i are reversed.

Let w be as in the hypothesis; we want to show

$$\|w\|_{L^{\bar{p}}(\Omega)} \leq K_6 \|g\|_{L^{\bar{p}}(\Omega)} \tag{47}$$

with $K_6 = K_5$, $1/\bar{p} + 1/\bar{q} = 1$, K_5, $\bar{q}$ as in Lemma 5. From well known results (see e.g [1]) we have

$$\|w\|_{L^{\bar{p}}(\Omega)} = \sup\left\{ \int_\Omega wf\, dx : f \in L^{\bar{q}}(\Omega),\ \|f\|_{L^{\bar{q}}(\Omega)} \leq 1 \right\}. \tag{48}$$

Let $f \in L^q(\Omega)$ $\forall q \geq 2$. Consider the Dirichlet problem

$$\begin{cases} a'(u, v) = \int_\Omega fv\, dx\ \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega) \end{cases} \tag{49}$$

The solution u is unique by Lemma 5. Since Ω is supposed to be bounded, the Riesz-Fredholm theory is valid and uniqueness of u implies its existence. By applying again Lemma 5 to the solution u, we get the existence of a number $\bar{q} \geq 2$ such that

$$\|u\|_{L^{\bar{q}}(\Omega)} \leq K_5 \|f\|_{L^{\bar{q}}(\Omega)}. \tag{50}$$

From (45), (49) it clearly follows

$$a'(u, w) = \int_\Omega fw\, dx = \int_\Omega gu\, dx . \tag{51}$$

From (48), (51), Lemma 5 and Hölder's inequality we finally get

$$\|w\|_{L^{\bar{p}}(\Omega)} = \sup\left\{ \int_\Omega gu\, dx : f \in L^{\bar{q}}(\Omega),\ \|f\|_{L^{\bar{q}}(\Omega)} \leq 1 \right\} \leq \tag{52}$$

$$\leq \sup\{\|g\|_{L^{\bar{p}}(\Omega)} \|u\|_{L^{\bar{q}}(\Omega)} : f \in L^{\bar{q}}(\Omega),\ \|f\|_{L^{\bar{q}}(\Omega)} \leq 1\} \leq K_5 \|g\|_{L^{\bar{p}}(\Omega)}$$

which completes the proof. ■

The next result, in a similar form, was already used in [4].

LEMMA 6. – *Let $\alpha \in Lip(\overline{\Omega})$, $\alpha \geq \bar{c}$ ($\bar{c}$ positive constant) in Ω, and $u \in H^1(\Omega)$ be a solution of the equation*

$$a(u, v) = \int_\Omega \left\{ f_0 v + \sum_{i=i}^n f_i v_{x_i} \right\} dx \qquad \forall v \in H_0^1(\Omega)$$

(*where the bilinear form $a(\cdot, \cdot)$ is defined in* (8)). *Then the function αu is solution of* *the equation*

$$a^*(au, v) = \int_\Omega \left\{ \left(af_0 + \sum_{i=1}^n f_i \alpha_{x_i} \right) v + \sum_{i=1}^n af_i v_{x_i} \right\} dx \qquad \forall v \in H_0^1(\Omega),$$

where we define

$$a^*(u, v) := \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij}^* u_{x_i} v_{x_j} + \sum_{i=1}^n (b_i^* u_{x_i} v + d_i^* u v_{x_i}) + c^* uv \right\} dx, \tag{53}$$

$$a_{ij}^* := a_{ij} \quad (i, j = 1, 2, \ldots, n),$$

$$b_i^* := b_i + \sum_{j=1}^n a_{ij} \alpha_{x_j}/\alpha \qquad (i = 1, 2, \ldots, n),$$

$$d_i^* := d_i - \sum_{j=1}^n a_{ji} \alpha_{x_j}/\alpha \qquad (i = 1, 2, \ldots, n),$$

$$c^* := c - \sum_{i=1}^n (b_i - d_i) \alpha_{x_i}/\alpha - \sum_{i,j=1}^n a_{ij} \alpha_{x_i} \alpha_{x_j}/\alpha^2.$$

PROOF. – The proof can be left to the reader. ∎

3. – Main result.

THEOREM 1. – *Suppose that the bilinear form $a_0(\cdot, \cdot)$ (defined in (2)) satisfies the same hypotheses of Lemma 1 and that there exists $p > n$ such that $b_i \in X^p(\Omega)$ ($i = 1, 2, \ldots n$). Then the Dirichlet problem (1) has a solution u, satisfying (2).*

PROOF. – We partially follow the same procedure of [4], [5]. First of all, according to Lemma 4, it is sufficient to show that the Dirichlet problem

$$\begin{cases} a_0(u, v) = \int_\Omega fv\, dx \ \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega) \end{cases} \tag{54}$$

has a solution whenever f is given in $H_0^1(\Omega)$ or, more generally, in $L^2(\Omega)$; this in turn is equivalent to show the a priori inequality

$$\|u\|_{L^2(\Omega)} \leq K_7 \|f\|_{L^2(\Omega)} \tag{55}$$

for the solution u of (54). If u is a solution of (54) and $f \in L^\infty(\Omega)$, we know that

$$\|u\|_{L^\infty(\Omega)} \leq (1/c_0) \|f\|_{L^\infty(\Omega)} \tag{56}$$

therefore it would be sufficient to prove an inequality such as

$$\|u\|_{L^1(\Omega)} \leq K_8 \|f\|_{L^1(\Omega)} \tag{57}$$

in order to get (55) by interpolation. Using again a duality argument, we remark that (57) is equivalent to

$$\|w\|_{L^\infty(\Omega)} \leq K_9 \|g\|_{L^\infty(\Omega)} \tag{58}$$

where $w \in H_0^1(\Omega)$ is the solution of the dual problem

$$\begin{aligned}(59)\quad & a_0'(w, v) := a_0(v, w) = \\ & = \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij} w_{x_j} v_{x_i} + \sum_{i=1}^n b_i w v_{x_i} + cwv \right\} dx = \int_\Omega gv\, dx \qquad \forall v \in H_0^1(\Omega).\end{aligned}$$

We also observe that, by the same duality arguments as above, the inequality

$$\|w\|_{L^1(\Omega)} \leq (1/c_0) \|g\|_{L^1(\Omega)} \tag{60}$$

holds, since it follows from (56). Finally, as in [4], [5] without loss of generality we can suppose Ω to be bounded, provided we prove that all the constants in the a priori inequalities are independent on Ω.

By using the above lemmata, we prove (58) as follows. Let $\{Q_h\}_{h \in \mathbb{N}}$ be a family of cubes of constant side length $r = 1$ which cover $\mathbb{R}^n$ as in Lemma 3;

let $\phi_h := \chi_{Q_h}$ $(h = 1, 2, \ldots)$, so that $\sum_{h=1}^{+\infty} \phi_h(x) = 1$ a.e. in $\mathbb{R}^n$. Let g be a given function in $L^\infty(\Omega)$ and consider the solution w_h of the Dirichlet problem

$$\begin{cases} a_0'(w_h, v) = \int_\Omega \phi_h g v\, dx \qquad \forall v \in H_0^1(\Omega), \\ w_h \in H_0^1(\Omega). \end{cases} \tag{61}$$

Since ϕ_h has compact support and $g \in L^\infty(\Omega)$, obviously $\phi_h g \in L^q(\Omega)$ for all $q \geq 1$, therefore from (60) it follows

$$\|w_h\|_{L^1(\Omega)} \leq (1/c_0) \|\phi_h g\|_{L^1(\Omega)}. \tag{62}$$

From (62) and the results of [2] (see Remark 4 in particular) we easily deduce

$$\|w_h\|_{L^\infty(\Omega)} \leq K_{10} \|\phi_h g\|_{L^\infty(\Omega)} \tag{63}$$

(note that $\|\phi_h g\|_{L^1(\Omega)} \leq \|\phi_h g\|_{L^\infty(\Omega)}$). Inequality (63) has the same form as (58), so by the interpolation argument above we have, for the time being, existence and uniqueness of the solution w_h of problem (61), and this is true for any $h \in \mathbb{N}$.

Notice also that it turns out $\sum_{h=1}^{+\infty} w_h = w$ because $\sum_{h=1}^{+\infty} \phi_h g = g$ in Ω and because of uniqueness which follows from (60). (As a matter of fact, since we have temporarily supposed Ω to be bounded, the sums with respect to h are finite, so $\sum_h w_h$ obviously belongs to $H_0^1(\Omega)$). From (63) the a priori inequality for w in $L^\infty(\Omega)$ would follow, but the constant would be dependent on Ω (more precisely, on the maximum value of $h \in \mathbb{N}$ such that $Q_h \cap \Omega \neq \emptyset$). Therefore a different argument must be used, as in [4], [5].

Let x_h be the center of the cube Q_h (for $h = 1, 2, \ldots$) and μ a positive constant; define $\alpha_h(x) := e^{\mu|x - x_h|}$. According to Lemma 6, the function $\alpha_h w_h$ satisfies the equation

$$a^*(\alpha_h w_h, v) = \int_\Omega \alpha_h \phi_h g v\, dx \qquad \forall v \in H_0^1(\Omega) \tag{64}$$

where the bilinear form $a^*(.,.)$ has coefficients

$$a_{ij}^* := a_{ji}\ (i, j = 1, 2, , \ldots, n),$$

$$b_i^* := \mu \sum_{j=1}^n a_{ji}(x_j - x_{hj})/|x - x_h|, \ (i = 1, 2, \ldots, n)$$

$$d_i^* := b_i - \mu \sum_{j=1}^n a_{ij}(x_j - x_{hj})/|x - x_h|, \ (i = 1, 2, \ldots, n)$$

$$c^* := c + \mu \sum_{i=1}^n b_i(x_i - x_{hi})/|x - x_h| - \mu^2 \sum_{i,j=1}^n a_{ij}(x_i - x_{hi})(x_j - x_{hj})/|x - x_h|^2$$

From the expressions of these coefficients and Lemma 5′, we can choose $\mu > 0$ so small that Lemma 5′ can be applied: therefore we deduce the following a priori inequality for the function $\alpha_h w_h$:

$$\|\alpha_h w_h\|_{L^{\bar p}(\Omega)} \leq K_6 \|\alpha_h \phi_h g\|_{L^{\bar p}(\Omega)} \qquad (h = 1, 2, \ldots) \tag{65}$$

for some $\bar p \geq 1$. Furthermore, obviously

$$\|\alpha_h \phi_h g\|_{L^{\bar p}(\Omega)} \leq K_{11} \|g\|_{L^\infty(\Omega)} \tag{66}$$

where the constant K_{11} depends only on n and μ. So by applying the results of [2] we deduce

$$\|\alpha_h w_h\|_{L^\infty(\Omega)} \leq K_{12}\Big[\|\alpha_h w_h\|_{L^{\bar p}(\Omega)} + \|\alpha_h \phi_h g\|_{L^{\bar p}(\Omega)}\Big]. \tag{67}$$

From the above inequalities and the definition of α_h it follows

$$|w_h(x)| \leq K_{13} e^{-\mu|x - x_h|} \|g\|_{L^\infty(\Omega)} \qquad \text{a.e. in } \Omega \ (h = 1, 2, \ldots) \tag{68}$$

whence

$$|w(x)| \leq \sum_{h=1}^\infty |w_h(x)| \leq K_{13} \|g\|_{L^\infty(\Omega)} \sum_{h=1}^{+\infty} e^{-\mu|x - x_h|} \qquad \text{a.e. in } \Omega. \tag{69}$$

Since the series on the right hand side converges, (58) is proved and the assertion follows as explained before. ■

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