HÖLDER REGULARITY FOR SOLUTIONS OF MIXED BOUNDARY VALUE PROBLEMS CONTAINING BOUNDARY TERMS

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Abstract. We prove Hölder regularity for solutions of mixed boundary value problems for a class of divergence form elliptic equations with discontinuous and unbounded coefficients, in the presence of boundary integrals.

1. Introduction.

In this note we want to study the regularity, on the boundary of Ω, of the solutions of a mixed problem for a class of divergence form elliptic equations, containing integral terms on the boundary.

In particular, given an open set Ω of $\mathbb{R}^n$, let us consider the subspace V of $H^1(\Omega)$ defined by

$$V := \{v \in H^1(\Omega) : \ v = 0 \text{ on } \Gamma_o \text{ in the sense of } H^1(\Omega)\} \tag{1}$$

where Γ_o is a closed (possibly empty) subset of $\partial\Omega$, and consider the bilinear form

$$a(u,v) := \int_\Omega \{\sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n (b_i u_{x_i} v + d_i u v_{x_i}) + cuv\}\, dx + \int_\Gamma guv\, d\sigma \tag{2}$$

where $\Gamma := \partial\Omega \backslash \Gamma_o$.

Let $u \in V$ be a solution of the equation

$$a(u,v) = \int_\Omega \{f_o v + \sum_{i=1}^n f_i v_{x_i}\}\, dx + \int_\Gamma hv\, d\sigma \quad \forall v \in V. \tag{3}$$

We can note that, if the functions we consider are sufficiently regular (for example of class $C^1(\overline{\Omega})$), as well as Γ, then u is a solution of the following problem:

$$\begin{cases} Lu = f_o - \sum_{i=1}^n (f_i)_{x_i} \quad \text{in } \Omega, \\ \sum_{i,j=1}^n a_{ij} u_{x_i} N_j + \sum_{i=1}^n d_i N_i u + gu = h + \sum_{i=1}^n f_i N_i \ \text{ on } \Gamma, \\ u = 0 \quad \text{on } \Gamma_o \end{cases} \tag{3'}$$

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where N is the normal unit vector to Γ (oriented towards the exterior of Ω) and L the operator defined by

$$(4)\qquad Lu := -\sum_{i,j=1}^{n} a_{ij}u_{x_ix_j} + \sum_{i=1}^{n}[b_i - d_i - \sum_{j=1}^{n}(a_{ij})_{x_j}]u_{x_i} + [c - \sum_{i=1}^{n}(d_i)_{x_i}]u$$

In a former work [5] we had supposed Ω possibly unbounded and studied minimal hypotheses on the coefficients of the bilinear form $a(.,.)$ and known terms f_i ($i = 0,1,2,\dots,n$) and h in order to obtain the boundedness of the same bilinear form on $V \times V$ and a priori inequalities in $L^\infty(\Omega)$ for the solutions of the boundary value problem

$$(5)\qquad \begin{cases} a(u,v) + \lambda(u,v)_{L^2(\Omega)} = \int_\Omega \{f_o v + \sum_{i=1}^{n} f_i v_{x_i}\}\,dx + \int_\Gamma hv\,d\sigma \qquad \forall v \in V, \\ u \in V. \end{cases}$$

In this note we want to study the regularity of solutions in a neighborhood of $\partial\Omega$. In fact, it is well known that, under suitable hypotheses on the coefficients of the bilinear form $a(.,.)$ and the data, a solution $u \in V$ of the equation (3) is Hölder continuous in the interior of Ω: see the classical results by De Giorgi [6], later extended by Stampacchia [17], [18], Moser [14], Ladyzhenskaya–Ural'tseva [11], Landis [12] and others.

In particular, the regularity of the solutions of a mixed problem has been studied for example by Fiorenza [7], Novruzov [15], Ibragimov [10], Pacella and Tricarico [16],..., but in all the works we know there are no integral terms on Γ. In the present note we prove the Hölder continuity of the solutions of the equation (3) also on Γ and on $\overline{\Gamma} \cap \Gamma_o$, under suitable hypotheses on the coefficients of the bilinear form $a(.,.)$, on the data and on the regularity of the set Γ.

In the proofs, we shall follow mainly Stampacchia [18]; therefore, for brevity, we shall report in detail only the new parts or the differences with respect to this paper.

2. Notations and hypotheses.

Let Ω be an open subset of $\mathbb{R}^n$ (with $n \geq 3$ for simplicity); since the regularity of solutions is a local property, it is not a restriction to suppose Ω bounded.

We remark that, under such hypothesis, the spaces $X^p(\Omega)$, $X^p_o(\Omega)$, defined in [3], both coincide with $L^p(\Omega)$. For the definition of the spaces $H^{1,p}(\Omega)$ we refer for example to [8], [11].

In $H^1(\Omega) := H^{1,2}(\Omega)$ we put by definition

$$||u_x||_{L^2(\Omega)} := \left\{ \sum_{i=1}^{n} ||u_{x_i}||^2_{L^2(\Omega)} \right\}^{1/2}$$

and assume as a norm for example the quantity

$$||u||_{H^1(\Omega)} := \left\{ ||u||^2_{L^2(\Omega)} + ||u_x||^2_{L^2(\Omega)} \right\}^{1/2}.$$

Now let us suppose $a_{ij} \in L^\infty(\Omega)$ $(i, j = 1, 2, \ldots, n)$, $\sum a_{ij} t_i t_j \geq \nu |t|^2$ $\forall t \in \mathbb{R}^n$ a.e. in Ω, with ν a positive constant. Except for further hypotheses, we shall suppose furthermore that $b_i \in L^n(\Omega)$, $d_i \in L^p(\Omega)$, $(i = 1, 2, \ldots, n)$, $c \in L^{p/2}(\Omega)$, $g \in L^{\overline{p}}(\Gamma)$ with $p > n$, $\overline{p} := p(n-1)/n$.

If $u \in H^1(\Omega)$, $m \in \mathbb{R}$, B is a closed subset of $\overline{\Omega}$, we shall say that $u \leq m$ $(u = m)$ on B in the sense of $H^1(\Omega)$ if there exists a sequence $u_j \in C^1(\overline{\Omega}) \cap H^1(\Omega)$ $(j = 1, 2, \ldots)$ such that $u_j \leq m$ $(u_j = m)$ in B for any $j \in \mathbb{N}$ and $\lim_j ||u - u_j||_{H^1(\Omega)} = 0$.

Let Γ_o be a closed (possibly empty) subset of $\partial\Omega$, and define $\Gamma := (\partial\Omega)\backslash\Gamma$. If $\overline{x} \in \mathbb{R}^n$ and $r > 0$, denote by $Q(\overline{x}, r)$ the open cube with center $\overline{x}$ and edge $2r$:

$$Q(\overline{x}, r) := \{x \in \mathbb{R}^n : \ |x_i - \overline{x}_i| < r \ (i = 1, 2, \ldots, n)\}$$

Furthermore let us denote

$$\Omega(\overline{x}, r) := \Omega \cap Q(\overline{x}, r), \qquad \Gamma(\overline{x}, r) := \Gamma \cap Q(\overline{x}, r).$$

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3. Hypotheses on the boundary of Ω.

In the present note we do not study the regularity of the solution on Γ_o (the part of $\partial\Omega$ where Dirichlet boundary condition is given), since this problem was already studied e.g. by Stampacchia [18], Gariepy e Ziemer [9], Maz'ya [13], Chicco [2] and others. We suppose that Γ is "locally Lipschitz continuous" in the following sense.

Let Ω_1 be an open subset of $\mathbb{R}^n$ such that $\Omega \subset \Omega_1$, $\overline{\Gamma} = (\partial\Omega_1) \cap (\partial\Omega)$, and therefore $\Gamma_o = \overline{(\partial\Omega)\backslash\Gamma}$. It is clear that the regularity of $\partial\Omega_1$ automatically implies a corresponding regularity of Γ.

Let us suppose that there exist two positive constants K, $\overline{r}$ such that, if $\overline{x} \in \partial\Omega_1$ and with

$$D := \{x \in \mathbb{R}^{n-1} : \ |x_i - \overline{x}_i| < \overline{r}, \ i = 1, 2, \ldots, n-1\} \tag{6}$$

$$Q(\overline{x}, \overline{r}) := \{x \in \mathbb{R}^n : \ |x - \overline{x}_i| < \overline{r}, \ i = 1, 2, \ldots, n\} \tag{7}$$

there exists a function $\phi : \ D \to \mathbb{R}$ such that

$$\phi(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_{n-1}) = \overline{x}_n, \tag{8}$$

$$Q(\overline{x}, \overline{r}) \cap \Omega_1 = \{x \in \mathbb{R}^n : \ (x_1, x_2, \ldots, x_{n-1}) \in D, \ x_n < \phi(x_1, x_2, \ldots, x_{n-1})\} \tag{9}$$

$$Q(\overline{x},\overline{r}) \cap (\partial\Omega_1) = \{x \in \mathbb{R}^n : \ (x_1, x_2, \ldots, x_{n-1}) \in D, \ x_n = \phi(x_1, x_2, \ldots, x_{n-1})\}$$

$$|\phi(x') - \phi(x'')| \leq K|x' - x''| \ \ \forall x', x'' \in D. \tag{10}$$

Consider now the hypotheses on $\Gamma_o \cap \overline{\Gamma}$. If a point $\overline{x} \in \Gamma_o \cap \overline{\Gamma}$ we suppose that it is possible to change the variables by a Lipschitz transformation (with inverse also Lipschitz), in such a way that both the following conditions a) and b) are satisfied:

a) there exists a positive number $\overline{r}$ such that

$$Q(\overline{x},\overline{r}) \cap \Omega \subset \{x \in \mathbb{R}^n : \ x_n < 0\}$$

$$Q(\overline{x},\overline{r}) \cap \overline{\Gamma} \subset \{x \in \mathbb{R}^n : \ x_n = 0\}$$

(this is a consequence of the preceding hypothesis on Γ);

b) there exist a positive number $\overline{r}$, a number p with $1 < p < n$ and a positive constant K_1 such that

$$||u||_{L^{p^*}(\Omega(\overline{x},\rho))} \leq K_1 ||u_x||_{L^p(\Omega(\overline{x},\rho))} \tag{11}$$

for every ρ with $0 < \rho < \overline{r}$ and every $u \in H^1(\Omega(\overline{x},\rho))$, $u = 0$ on $\Gamma_o \cap Q(\overline{x},\rho)$ in the sense of $H^1(\Omega(\overline{x},\rho))$.

We can remark that, when $\partial\Omega$ is very regular in a neighborhood of $\overline{x}$, as well as the $(n-2)-$dimensional manifold $\Gamma_o \cap \overline{\Gamma}$, we can assume (eventually after a suitable change of variables by Lipschitz functions) that

$$\begin{gathered} Q(\overline{x},\overline{r}) \cap \Gamma_o \cap \overline{\Gamma} \subset \{x \in \mathbb{R}^n : \ x_n = x_{n-1} = 0\} \\ Q(\overline{x},\overline{r}) \cap \Gamma_o \subset \{x \in \mathbb{R}^n : \ x_n \leq 0, \ x_{n-1} \leq 0\} \\ Q(\overline{x},\overline{r}) \cap \overline{\Gamma} \subset \{x \in \mathbb{R}^n : \ x_n = 0, \ x_{n-1} \geq 0\} \end{gathered}$$

In this case, by proceeding as in [17] and remembering the results of [18] and [2], it is possible to verify that property b) above is satisfied.

4. Preliminary results.

In the present paragraph we extend some well known results in order to adapt them to our needs.

Lemma 1. *There exists a positive number r_o, depending only on the regularity of Γ, such that for every $\overline{x} \in \overline{\Gamma}$ there exists a cube Q with center $\overline{x}$ and edge $2r_o$ with the following properties. If $u \in H^1(\Omega \cap Q)$, $u = 0$ on $\partial(\Omega \cap Q) \backslash \Gamma$ in the sense of $H^1(\Omega \cap Q)$, we have*

$$||u||_{L^{2^*}(\Omega \cap Q)} \leq K_2 ||u_x||_{L^2(\Omega \cap Q)} \tag{12}$$

where K_2 is a constant depending only on n and K (where K is the Lipschitz constant of Γ: see (10)) and $2^* := 2n/(n-2)$.

Proof. By the results of [5], there exists a positive number $\overline{r}$, depending only on Γ, such that if

$$\delta := \min\{1/2,\ 1/(2K\sqrt{n-1})\} \tag{13}$$

$$Q_\delta(\overline{x}, \overline{r}) := \{x \in \mathbb{R}^n :\ |x_i - \overline{x}_i| < \delta\overline{r}\ (i = 1, 2, \ldots, n-1),\ |x_n - \overline{x}_n| < \overline{r}\} \tag{14}$$

the set $Q_\delta(\overline{x}, \overline{r}) \cap \Omega$ is converted, by the change of variables

$$\begin{cases} y_i = (x_i - \overline{x}_i)/\delta\ (i = 1, 2, \ldots, n-1) \\ y_n = 2\overline{r} - 2\overline{r}(x_n - \overline{x}_n + \overline{r})/[\phi(x_1, x_2, \ldots, x_{n-1}) - \overline{x}_n + \overline{r}] \end{cases} \tag{15}$$

into the cube

$$\widetilde{Q}(o, \overline{r}) := \{y \in \mathbb{R}^n :\ |y_i| < \overline{r}\ (i = 1, 2, \cdots, n-1), 0 < y_n < 2\overline{r}\} \tag{16}$$

Consider now the cube $Q(\overline{x}, \delta\overline{r})$. It turns out simply (see (18) in [5])

$$Q(\overline{x}, \delta\overline{r}) \subset Q_\delta(\overline{x}, \overline{r}) \subset Q(\overline{x}, \overline{r}) \tag{17}$$

If $u \in H^1(\Omega(\overline{x}, \delta\overline{r})),\ u = 0$ on $\partial(Q(\overline{x}, \delta\overline{r}))\backslash\Gamma$ (in the sense of $H^1(\Omega(\overline{x}, \delta\overline{r}))$), we can extend the definition of u in $Q_\delta(\overline{x}, \overline{r}) \cap \Omega \backslash Q(\overline{x}, \delta\overline{r})$ by defining it equal to zero there, in such a way that, denoting again the function so extended by u, we have
$u \in H^1(Q_\delta(\overline{x}, \overline{r}) \cap \Omega)$ and

$$||u||_{H^1(Q_\delta(\overline{x}, \overline{r}) \cap \Omega)} = ||u||_{H^1(\Omega(\overline{x}, \delta\overline{r}))} \tag{18}$$

From our hypotheses, the function $\widetilde{u}$ (obtained by transforming u through the change of variables) is zero on all the faces of $\widetilde{Q}$ except the one corresponding to $\Gamma \cap Q$, i.e. where $y_n = 0$.

Let us consider now the parallelepiped

$$P := \{y \in \mathbb{R}^n :\ |y_i| < \overline{r}\ (i = 1, 2, \ldots, n-1),\ |y_n| < 2\overline{r}\} \tag{19}$$

in which we extend the definition of the function $\widetilde{u}$ by putting, for $-2\overline{r} < y_n \leq 0$:

$$\widetilde{u}(y_1, y_2, \ldots, y_n) := \widetilde{u}(y_1, y_2, \ldots, -y_n) \tag{20}$$

(that is we extend $\widetilde{u}$ as an "even function" with respect to the variable y_n). From the theory of Sobolev spaces and our hypotheses it follows that the function $\widetilde{u}$, as extended by (20), belongs to $H^1_o(P)$, therefore from known results it turns out

$$||\widetilde{u}||_{L^{2^*}(P)} \leq K_3||\widetilde{u}_x||_{L^2(P)} \tag{21}$$

where K_3 depends only n (see e.g. [8]). From (20), (21) we deduce easily that

$$||\widetilde{u}||_{L^{2^*}(\widetilde{Q})} \leq K_3||\widetilde{u}_x||_{L^2(\widetilde{Q})} \tag{22}$$

and finally, by applying the change of variables inverse of (15), we deduce the inequality

$$||u||_{L^{2^*}(Q_\delta(\overline{x}, \overline{r}) \cap \Omega)} \leq K_2||u_x||_{L^2(Q_\delta(\overline{x}, \overline{r}) \cap \Omega)} \tag{23}$$

where, as we have seen, the constant K_2 depends on n and K, Lipschitz constant of the function which represents Γ in a neighborhood of $\overline{x}$. From (23), remembering (18), we get the conclusion, where we choose $r_o := \delta\overline{r}$ and $Q := Q(\overline{x}, \delta\overline{r})$. □

Lemma 2. *There exists a positive number $\overline{r}$ depending only on the regularity of Γ, such that if $\overline{x} \in \overline{\Gamma}$ we can find a cube Q with center $\overline{x}$ and edge $2\overline{r}$ having the following properties. If $u \in H^{1,s}(\Omega \cap Q)$, $u = 0$ on $\partial(\Omega \cap Q) \backslash \Gamma$ in the sense of $H^{1,s}(\Omega \cap Q)$, with $1 < s < n$, we have*

$$||u||_{L^{s(n-1)/(n-s)}(\Gamma\cap Q)} \leq K_4 ||u_x||_{L^s(\Omega\cap Q)} \tag{24}$$

where K_4 is a constant depending only on s, n and K (Lipschitz constant of Γ: see (10)).

Proof. Proceeding in a similar way to the preceding lemma, through a Lipschitz change of variables (with an inverse Lipschitz also) we can consider only the case in which the function $\widetilde{u}$ (obtained from u by the variable transformation) is defined in the cube $\widetilde{Q}$ (see (16)), where $\widetilde{u} = 0$ on all the faces of the cube except (eventually) the one where $y_n = 0$. From (36) of [5] we deduce

$$\begin{aligned} &||\widetilde{u}||_{L^{s(n-1)/(n-s)}((\partial\widetilde{Q})\cap\{y:\ y_n=0\})} \leq \\ &\leq \{1 + (n-1)K^2\}^{(n-s)/2s(n-1)} K_5 \left[(1/\overline{r})||\widetilde{u}||_{L^s(\widetilde{Q})} + ||\widetilde{u}_y||_{L^s(\widetilde{Q})}\right] \end{aligned} \tag{25}$$

where the constant K_5 depends only on s and n. Now remark that if instead of the cube $\widetilde{Q}$ defined by (16) we consider the new cube

$$\widetilde{Q}_\lambda := \{y \in \mathbb{R}^n :\ |y_i| < \lambda\overline{r}(i = 1, 2, \ldots, n-1),\ 0 < y_n < 2\lambda\overline{r}\} \tag{26}$$

with $\lambda > 1$ constant, the function $\widetilde{u}$, extended equal to zero in $\widetilde{Q}_\lambda \backslash \widetilde{Q}$, clearly belongs to $H^{1,s}(\widetilde{Q}_\lambda)$. Therefore we can rewrite (25) with $\lambda\overline{r}$ instead of $\overline{r}$, i.e.

$$\begin{aligned} &||\widetilde{u}||_{L^{s(n-1)/(n-s)}((\partial\widetilde{Q}_\lambda)\cap\{y:\ y_n=0\})} \leq \\ &\leq \{1 + (n-1)K^2\}^{(n-s)/2s(n-1)} K_5 \left[(1/\lambda\overline{r})||\widetilde{u}||_{L^s(\widetilde{Q}_\lambda)} + ||\widetilde{u}_y||_{L^s(\widetilde{Q}_\lambda)}\right] \end{aligned} \tag{27}$$

from which, by letting λ tend to infinity, we deduce

$$\begin{aligned} &||\widetilde{u}||_{L^{s(n-1)/(n-s)}((\partial\widetilde{Q})\cap\{y:\ y_n=0\})} \leq \\ &\qquad \leq \{1 + (n-1)K^2\}^{(n-s)/2s(n-1)} K_5 ||\widetilde{u}_y||_{L^s(\widetilde{Q})} \end{aligned} \tag{28}$$

Finally, by applying the change of variables inverse of the one we used before, from (28) we easily arrive at the conclusion. □

Theorem 1. *Consider the bilinear form*

$$a(u,v) := \int_\Omega \{\sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n (b_i u_{x_i} v + d_i u v_{x_i}) + cuv\}\, dx + \int_\Gamma guv\, d\sigma$$

in which we assume $b_i, d_i \in L^n(\Omega)$ $(i = 1, 2, \ldots, n)$, $c \in L^{n/2}(\Omega)$, $g \in L^{n-1}(\Gamma)$. Then there exists a positive number $\overline{r}$ such that, if Q is a cube with edge $2r \leq 2\overline{r}$ and center $\overline{x} \in \overline{\Gamma}$, the bilinear form $a(.,.)$ is coercive on

$$V_Q := \{v \in H^1(\Omega \cap Q) :\ v = 0 \text{ on } \partial(\Omega \cap Q) \backslash \Gamma \text{ in the sense of } H^1(\Omega \cap Q)\}$$

Proof. We must prove that there exists a positive constant K_6, depending on the coefficients of the bilinear form $a(.,.)$, on n and on K (Lipschitz constant of the function which reperesents locally Γ), such that

$$a(v,v) \geq K_6 ||v||^2_{H^1(\Omega\cap Q)} \qquad \forall v \in V_Q. \tag{29}$$

as soon as Q is chosen as explained above.

This inequality can be easily obtained remembering the hypotheses on the coefficients and lemmata 1 and 2. In fact, let us choose the positive number $\overline{r}$ so small that lemmata 1 and 2 are applicable for the cube $Q := \{x \in \mathbb{R}^n : \ |x_i - \overline{x}_i| < \overline{r} \ (i = 1, 2, \dots, n)\}$. By taking into account also Hölder's inequality we have

$$\left| \sum_{1=1}^{n} \int_{\Omega\cap Q} b_i v_{x_i} v \, dx \right| \leq \sum_{i=1}^{n} ||b_i||_{L^n(\Omega\cap Q)} ||v_x||_{L^2(\Omega\cap Q)} ||v||_{L^{2^*}(\Omega\cap Q)} \leq \tag{30}$$

$$\leq K_2 \sum_{i=1}^{n} ||b_i||_{L^n(\Omega\cap Q)} ||v_x||^2_{L^2(\Omega\cap Q)}$$

(31)

$$\left| \sum_{1=1}^{n} \int_{\Omega\cap Q} d_i v_{x_i} v \, dx \right| \leq \sum_{i=1}^{n} ||d_i||_{L^n(\Omega\cap Q)} ||v_x||_{L^2(\Omega\cap Q)} ||v||_{L^{2^*}(\Omega\cap Q)} \leq$$

$$\leq K_2 \sum_{i=1}^{n} ||d_i||_{L^n(\Omega\cap Q)} ||v_x||^2_{L^2(\Omega\cap Q)}$$

$$\left| \int_{\Omega\cap Q} c v^2 \, dx \right| \leq ||c||_{L^{n/2}(\Omega\cap Q)} ||v||^2_{L^{2^*}(\Omega\cap Q)} \leq \tag{32}$$

$$\leq K_2^2 ||c||_{L^{n/2}(\Omega\cap Q)} ||v_x||^2_{L^2(\Omega\cap Q)}$$

while from lemma 2 with $s = 2$ we have

$$\left| \int_{\Gamma\cap Q} g v^2 \, d\sigma \right| \leq K_4^2 ||g||_{L^{n-1}(\Gamma\cap Q)} ||v_x||^2_{L^2(\Omega\cap Q)} \tag{33}$$

From our hypotheses on the functions $b_i, \ d_i, \ c, \ g$ it follows easily that there exists a positive number $\overline{r}$ (depending on these coefficients) such that, even satisfying the preceding choice, if $0 < r \leq \overline{r}$ and if the cube Q, with centre $\overline{x} \in \overline{\Gamma}$, has edge $2r$, we have

$$K_2 \left(\sum_{i=1}^{n} ||b_i||_{L^n(\Omega\cap Q)} + \sum_{i=1}^{n} ||d_i||_{L^n(\Omega\cap Q)} + K_2 ||c||_{L^{n/2}(\Omega\cap Q)} \right) + \tag{34}$$

$$+ K_4^2 ||g||_{L^{n-1}(\Gamma\cap Q)} \leq \nu/4$$

From (30), (31),..., (34), and taking into account the uniform ellipticity and lemma 1, we get the conclusion with $K_6 = \nu/2$. □

5. Local behavior of subsolutions.

In this paragraph we want to study how to apply to our situation the results of [18] in order to obtain some a priori inequality for the essential supremum of subsolutions in subsets of Ω with small measure.

Lemma 3. *There exist two positive constants $\overline{r}$, K_7, depending on n, Γ and the coefficients of the bilinear form $a(.,.)$, such that what follows is true. Let $\overline{x} \in \Gamma$, $u \in H^1(\Omega(\overline{x},\overline{r}))$, $u \leq 0$ on $\partial(\Omega(\overline{x},\overline{r}))\backslash\Gamma$,*

$$a(u,v) \leq \int_{\Omega(\overline{x},\overline{r})} \{f_o v + \sum_{i=1}^n f_i v_{x_i}\}\, dx + \int_{\Gamma(\overline{x},\overline{r})} hv\, d\sigma$$

for any $v \in H^1(\Omega(\overline{x},\overline{r}))$, $v \geq 0$ in $\Omega(\overline{x},\overline{r})$, $v = 0$ on $\partial(\Omega(\overline{x},\overline{r}))\backslash\Gamma$ in the sense of $H^1(\Omega(\overline{x},\overline{r}))$. Then if r is such that $0 < r \leq \overline{r}$ we have

(35)
$$\operatorname*{ess\,sup}_{\Omega(\overline{x},r)} u \leq$$
$$\leq K_7 \left[||f_o||_{L^{np/(n+p)}(\Omega(\overline{x},\overline{r}))} + \sum_{i=1}^n ||f_i||_{L^p(\Omega(\overline{x},\overline{r}))} + ||h||_{L^{\overline{p}}(\Gamma(\overline{x},\overline{r}))} \right] r^{1-n/p}$$

Proof. By a simple change of variables (dilation) we see that it is sufficient to prove the result when $r = \overline{r}$. For this purpose we choose $\overline{r}$ as in the preceding theorem in such a way that the bilinear form $a(.,.)$ is coercitive on

$V_Q := \{v \in H^1(\Omega \cap Q) : v = 0 \text{ on } \partial(\Omega \cap Q)\backslash\Gamma \text{ in the sense of } H^1(\Omega \cap Q)\}$
where Q is the cube with center $\overline{x}$ and edge $2\overline{r}$, that is

$$a(v,v) \geq K_6 ||v||^2_{H^1(\Omega\cap Q)} \qquad \forall v \in V_Q. \tag{36}$$

From theorem 3 of [5], where we put $m = 0$, we have

$$\operatorname*{ess\,sup}_{\Omega\cap Q} u \leq K_8 ||u^+||_{H^1(\Omega\cap Q)} + K_9 \tag{37}$$

where we have defined $u^+ := \max(u,0)$ and K_8, K_9 are the constants of [5]. From (36) with $v = u^+$ (allowable since $u^+ \in V_Q$) we deduce

$$||u^+||^2_{H^1(\Omega\cap Q)} \leq K_6^{-1} a(u^+,u^+) \tag{38}$$

whence, remembering that $a(u,u^+) = a(u^+,u^+)$, it follows
(39)
$$||u^+||^2_{H^1(\Omega\cap Q)} \leq K_6^{-1} \left[\int_{\Omega\cap Q} f_o u^+\, dx + \sum_{i=1}^n \int_{\Omega\cap Q} f_i (u^+)_{x_i}\, dx + \int_{\Gamma\cap Q} hu^+\, d\sigma \right]$$

From this inequality, remembering lemmata 1 and 2 (with $s = 2$) we easily get

(40)
$$||u^+||_{H^1(\Omega\cap Q)} \leq$$
$$\leq K_6^{-1} \left[K_2 ||f_o||_{L^{2n/(n+2)}(\Omega\cap Q)} + \sum_{i=1}^n ||f_i||_{L^2(\Omega\cap Q)} + K_4 ||h||_{L^{n-1}(\Gamma\cap Q)} \right]$$

From (37) and (40), remembering the results of [5] and that $p > n$, we reach the conclusion in the form

(41)
$$\operatorname*{ess\,sup}_{\Omega\cap Q} u \le$$
$$\le K_7 \left[||f_o||_{L^{np/(n+p)}(\Omega\cap Q)} + \sum_{i=1}^{n} ||f_i||_{L^p(\Omega\cap Q)} + ||h||_{L^{\overline{p}}(\Gamma\cap Q)} \right]$$

where, as we have said, $\overline{p} := p(n-1)/n$ (see [5]) and

$$Q := \{x \in \mathbb{R}^n : \ |x_i - \overline{x}_i| < \overline{r} \ (i = 1, 2, \dots, n)\}$$

Now let $0 < r \le \overline{r}$ and define

$$Q_r := \{x \in \mathbb{R}^n : \ |x_i - \overline{x}_i| < r \ (i = 1, 2, \dots, n)\}$$

then from (41) with a simple dilation we get

(42)
$$\operatorname*{ess\,sup}_{\Omega\cap Q_r} u \le$$
$$\le K_7 \left[||f_o||_{L^{np/(n+p)}(\Omega\cap Q)} + \sum_{i=1}^{n} ||f_i||_{L^p(\Omega\cap Q)} + ||h||_{L^{\overline{p}}(\Gamma\cap Q)} \right] r^{1-n/p}$$

where the constant K_7 depends on n, $\Gamma, \overline{r}$ and the coefficients of the bilinear form $a(.,.)$, but depends neither on u nor on r (as long as $0 < r \le \overline{r}$). The precise dependence of the constant K_7 on the coefficients of $a(.,.)$ may be easily deduced from the results of [5]. In fact, we have already remarked that, since Ω is supposed bounded, it turns out $X^p(\Omega) = X^p_o(\Omega) = L^p(\Omega)$. □

The preceding lemma gives an evaluation of the subsolutions (and therefore of the solutions) not positive on a part of the boundary of Ω. Nevertheless, proceeding like in [18], it is necessary also to find some local inequality in L^∞ without knowing the behavior of the solutions on the boundary of $\Omega \cap Q$ (except the fact of being zero on Γ_o). In other words, in similarity of theorem 5.5 of [18], it is useful to prove the following

Theorem 2. *There exists a positive number $\overline{r}$, depending on Γ and the coefficients of $a(.,.)$, such that if $\overline{x} \in \partial\Omega$ and $u \in H^1(\Omega(\overline{x},\overline{r}))$, $u = 0$ on $\Gamma_o \cap Q(\overline{x},\overline{r})$ is solution of the inequality*

$$a(u,v) \le \int_{\Omega(\overline{x},\overline{r})} \{f_o v + \sum_{i=1}^{n} f_i v_{x_i}\}\, dx + \int_{\Gamma(\overline{x},\overline{r})} hv \, d\sigma$$

for any $v \in H^1(\Omega(\overline{x},\overline{r}))$, $v \ge 0$ in $\Omega(\overline{x},\overline{r})$, $v = 0$ on $\partial(\Omega(\overline{x},\overline{r}))\backslash\Gamma$, and $r \le \overline{r}$, we have

(43)
$$\operatorname*{ess\,sup}_{\Omega\cap Q_{r/2}} u \le K_{10} \left[||f_o||_{L^{np/(n+p)}(\Omega\cap Q_r)} \right] r^{1-n/p} +$$
$$+ \left[\sum_{i=1}^{n} ||f_i||_{L^p(\Omega\cap Q_r)} + ||h||_{L^{\overline{p}}(\Gamma\cap Q_r)} + r^{-n/2} ||u||_{L^2(\Omega\cap Q_r)} \right] r^{1-n/p}$$

where we have defined for brevity $Q_r := Q(\overline{x}, r)$ *and* K_{10} *is a constant depending only on* n, Γ *and the coefficients of* $a(.,.)$.

Proof. The theorem is an extension of theorem 5.5 of [18] (and more precisely it coincides with it when $\Gamma_o \cap Q(\overline{x}, \overline{r}) = (\partial\Omega) \cap Q(\overline{x}, \overline{r})$). The proof also may follow that of [18], with obvious changes; for example the preceding lemma will be used instead of theorem 4.2 of [18]. On the other hand, theorems 5.1, 5.2, 5.3 of [18] and their corollaries are consequences of lemma 5.2 of [18] and Sobolev and Hölder inequalities; in conclusion it will be sufficient to prove the analogous of lemma 5.2 of [18], that is the following:

Lemma 4. *Let* $\overline{x} \in \partial\Omega$. *There exists a positive number* $\overline{r}$, *depending on* Γ *and the coefficients of* $a(.,.)$, *such that if* $u \in H^1(\Omega(\overline{x}, \overline{r}))$, $u \geq 0$ *in* $\Omega(\overline{x}, \overline{r})$, $u = 0$ *on* $\Gamma_o \cap Q(\overline{x}, \overline{r})$ *in the sense of* $H^1(\Omega(\overline{x}, \overline{r}))$, *and it turns out* $a(u, v) \leq 0$ *for all* $v \in H^1(\Omega(\overline{x}, \overline{r}))$, $v = 0$ *on* $\partial(\Omega(\overline{x}, \overline{r}))\backslash\Gamma$ *in the sense of* $H^1(\Omega(\overline{x}, \overline{r}))$, $v \geq 0$ *in* $\Omega(\overline{x}, \overline{r})$ *and furthermore* $\alpha \in C^1(\overline{\Omega(\overline{x}, \overline{r})})$, $\alpha = 0$ *on* $\partial(\Omega(\overline{x}, \overline{r}))\backslash\partial\Omega$, *we have*

$$\int_{\Omega(\overline{x},\overline{r})} \alpha^2 u_x^2 \, dx \leq K_{11} \int_{\Omega(\overline{x},\overline{r})} (\alpha^2 + \alpha_x^2) u^2 \, dx \tag{44}$$

where K_{11} *is a constant depending on* n, Γ, $\overline{r}$ *and the coefficients of* $a(.,.)$.

Proof. This lemma also can be proved in the same way as the corresponding lemma 5.2 of [18], by using inequalities (12), (24) instead of the usual theorems of Sobolev. For simplicity we shall treat only the integral on Γ. We have (see (38) in [5])

$$\begin{aligned} &\left| \int_{\Gamma(\overline{x},\overline{r})} g \alpha^2 u^2 \, d\sigma \right| \leq \\ &\leq K_{12} \omega(g, n-1, \sqrt{1 + (n-1)K^2} (2\overline{r})^{n-1}) \times \\ &\qquad \times \left[(1/\overline{r}^2) ||\alpha u||^2_{L^2(\Omega(\overline{x},\overline{r}))} + ||(\alpha u)_x||^2_{L^2(\Omega(\overline{x},\overline{r}))} \right] \end{aligned} \tag{45}$$

where K_{12} is a constant depending only on n and K. Therefore it is possible to determine $\overline{r}$ in such a way that

$$K_{12} \omega(g, n-1, \sqrt{1 + (n-1)K^2} (2\overline{r})^{n-1}) \leq \nu/16 \tag{46}$$

so from (45) we deduce

$$\begin{aligned} &\left| \int_{\Gamma(\overline{x},\overline{r})} g \alpha^2 u^2 \, d\sigma \right| \leq \\ &\leq K_{13} \left[||\alpha u||^2_{L^2(\Omega(\overline{x},\overline{r}))} + ||\alpha_x u||^2_{L^2(\Omega(\overline{x},\overline{r}))} \right] + (\nu/8) ||\alpha u_x||^2_{L^2(\Omega(\overline{x},\overline{r}))} \end{aligned} \tag{47}$$

where K_{13} is a constant depending on the same quantities of K_{12} and on $\overline{r}$. We remark that, from our hypotheses, the function $\alpha^2 u$ is non negative and equal to zero on $\partial(\Omega(\overline{x}, \overline{r}))\backslash\Gamma$ (in the sense of $\Omega(\overline{x}, \overline{r})$); therefore it can replace v as a test function in the inequality $a(u, v) \leq 0$. So we can proceed as in [18]; from (47) and from similar inequalities, obtained as in [18] it is easy to arrive at the conclusion. □

6. Regularity of subsolutions.

In the present paragraph we briefly describe the procedure that leads to the hölderness of solutions, under suitable hypotheses on the coefficients.

Theorem 3. *Let Ω be an open subset of $\mathbb{R}^n$ and suppose that the hypotheses on $\partial\Omega$ mentioned in paragraph 3 are satisfied. Let $u \in V$ be a solution of the equation*

$$a(u,v) = \int_\Omega \{f_o v + \sum_{i=1}^n f_i v_{x_i}\}\, dx + \int_\Gamma hv\, d\sigma \quad \forall v \in V \tag{48}$$

where we suppose that the coefficients of the bilinear form $a(.,.)$ satisfy the same hypotheses of paragraph 2, and furthermore $f_o \in L^{p/2}(\Omega)$, $f_i \in L^p(\Omega)$ $(i = 1, 2, \ldots, n)$, $h \in L^{\overline{p}}(\Gamma)$ with $p > n$, $\overline{p} := p(n-1)/n$. Finally let $\overline{x} \in \partial\Omega$. Then there exist three positive constants K_{14}, $\overline{r}$, λ (with $\lambda < 1$), depending on the coefficients of $a(,.,)$ and on $\partial\Omega$, such that

$$|u(x) - u(\overline{x})| \le K_{14}[||u||_{L^2(\Omega(\overline{x},\overline{r}))} + ||h||_{L^{\overline{p}}(\Gamma(\overline{x},\overline{r}))} + \\ + ||f_o||_{L^{np/(n+p)}(\Omega(\overline{x},\overline{r}))} + \sum_{i=1}^n ||f_i||_{L^p(\Omega(\overline{x},\overline{r}))}]|x - \overline{x}|^\lambda \tag{49}$$

for any $x \in \Omega(\overline{x},\overline{r})$

Proof. As in [18] the proof can be achieved through several steps:
1) by supposing temporarily $c = d_i = g = h = f_i = 0$ $(i = 0, 1, 2, \ldots, n)$;
2) by supposing still $c = d_i = g = 0$ $(i = 1, 2, \ldots, n)$ but letting h, f_i $(i = 0, 1, 2, \ldots, n)$ to be eventually non zero;
3) considering the general case.

Let us begin by supposing $c = d_i = g = h = f_i = 0$ $(i = 0, 1, 2, \ldots, n)$. If $\overline{x} \in \Gamma_o \backslash \overline{\Gamma}$, the result is known (see for example [18]). So let us suppose $\overline{x} \in \overline{\Gamma}$. If $\overline{x} \in \Gamma$, one can choose a number $\overline{r} > 0$ such that $\overline{Q(\overline{x},\overline{r}) \cap \Gamma} \subset \Gamma$ and that the set $Q(\overline{x},\overline{r}) \cap \Omega$ can be transformed in a parallelepiped P by a change of variables with a Lipschitz function, having inverse function also Lipschitz (please note that a similar operation has already been made in lemmata 1 and 2, and is possible because of our hypotheses on Γ). More precisely, let us suppose that, after the change of variables, the point $\overline{x}$ coincide with the origin of the coordinates and it turns out

$$\Omega \cap Q = \{x \in \mathbb{R}^n : \; |x_i| < \overline{r} \; (i = 1, 2, \ldots, n-1), \; -\overline{r} < x_n < 0\} \tag{50}$$

So, proceeding as in [1], we can extend the definition of the function u as an "even function" with respect to the variable x_n, that is by putting

$$u(x_1, x_2, \ldots, x_n) := u(x_1, x_2, \ldots, -x_n) \\ \text{for } |x_i| < \overline{r} \; (i = 1, 2, \ldots, n-1), \; 0 < x_n < \overline{r}\} \tag{51}$$

in such a way that the function u is defined in all the cube

$$\widehat{Q} := \{x \in \mathbb{R}^n : \ |x_i| < \overline{r} \ (i = 1, 2, \dots, n)\}$$

from known properties of Sobolev spaces, we can prove that the function u, extended as before, belongs to $H^1(\widehat{Q})$ and is a solution of the equation

$$a(u, v) = 0 \qquad \forall v \in H^1_o(\widehat{Q}) \tag{52}$$

provided we extend the definition of the coefficients of the bilinear form $a(.,.)$ to all $\widehat{Q}$ in a suitable way, as in [1]. By this procedure we get the Hölder continuity of the function u, since it is a solution of the equation (52), and applying the results (for example) of [18].

Now let us suppose $\overline{x} \in \overline{\Gamma} \cap \Gamma_o$; from our hypotheses, by means of a change of Lipschitz continuous variables and having an inverse also Lipschitz continuous, we can suppose that $\overline{x}$ coincides with the origin of coordinates and

$$Q(o, \overline{r}) \cap \Omega \subset \{x \in \mathbb{R}^n : \ x_n < 0\}$$

$$\Gamma \cap Q(o, \overline{r}) \subset \{x \in \mathbb{R}^n : \ x_n = 0\}$$

By hypothesis, also condition b) of paragraph 2 (inequality (11)) is valid.

First of all let us extend the definition of u to all of $Q \cap \{x \in \mathbb{R}^n : \ x_n < 0\}$ by putting $u(x) = 0$ if $x \in Q \cap \{x \in \mathbb{R}^n : \ x_n < 0\} \backslash \Omega$. For our previous hypotheses it turns out $(\partial\Omega) \cap \overline{Q} \cap \{x \in \mathbb{R}^n : \ x_n < 0\} \subset \Gamma_o$ hence it follows that the function u extended in this way belongs to $H^1(Q \cap \{x \in \mathbb{R}^n : \ x_n < 0\})$.

Furthermore, let us extend the definition of the function u to all of Q by putting, as in (51),

(53)

$$u(x_1, x_2, \dots, x_n) := u(x_1, x_2, \dots, -x_n)$$
$$\text{when } |x_i| < \overline{r} \ (i = 1, 2, \dots, n-1), \ 0 < x_n < \overline{r}$$

then the function u, extended in this way, clearly belongs to $H^1(Q)$. Let us define also

$$A := \{x \in \mathbb{R}^n : \ (x_1, x_2, \dots, -x_n) \in \Omega \cap Q\}$$

$$\Omega^* := \text{interior of}(\Omega \cup \Gamma \cup A) \cap Q$$

So, from hypothesis b) of paragraph 2 (formula (11)), it follows:

$$||u||_{L^{p^*}(\Omega^*(o,\rho))} \leq K_1 ||u_x||_{L^p(\Omega^*(o,\rho))} \tag{54}$$

for any ρ with $0 < \rho < \overline{r}$, where we have defined

$$\Omega^*(o, \rho) := \Omega^* \cap Q(o, \rho)$$

The function u, as extended by (53), is evidently zero on $(\partial\Omega^*) \cap Q(o,\rho)$ (with $0 < \rho < \overline{r}$), and is solution, in Ω^*, of the equation

$$a(u,v) = 0 \qquad \forall v \in H^1_o(\Omega^*) \tag{55}$$

after extending the coefficients of the bilinear form $a(.,.)$ to $\Omega^* \backslash \Omega$ as in [1]. Therefore, by taking into account (55) and the results of [2], [18],... we deduce again the Hölder regularity of the solution u in o. The case $c = d_i = g = h = f_i = 0$ $(i = 0,1,\ldots,n)$ is completely proved.

2) Now let us suppose, following again Stampacchia [18], that $c = g = d_i = 0$ $(i = 1,2,\ldots,n)$ but $h,\ f_i$ $(i = 0,1,2,\ldots,n)$ not necessarily zero. Let us fix $\overline{r}$ as in 1), and consider the solution v of the boundary value problem

$$\begin{cases} a(v,\phi) = \displaystyle\int_\Omega \{f_o\phi + \sum_{i=1}^n f_i\phi_{x_i}\}\,dx + \int_\Gamma h\phi\,d\sigma \quad \forall\phi \in V_{\overline{r}} \\ v \in V_{\overline{r}} \end{cases} \tag{56}$$

where we have defined

$$V_{\overline{r}} := \{\phi \in H^1(\Omega(\overline{x},\overline{r})) :\ \phi = 0 \text{ on } (\partial\Omega(\overline{x},\overline{r}))\backslash\Gamma \text{ in the sense of } H^1(\Omega(\overline{x},\overline{r}))\}$$

Since the bilinear form $a(.,.)$ is coercitive on $V_{\overline{r}}$ (for the choice of $\overline{r}$, see theorem 1), the problem (56) has one and only one solution v. If we define $w := u - v$, it clearly turns out that w is a solution of the equation

$$a(w,\phi) = 0 \qquad \forall\phi \in V_{\overline{r}} \tag{57}$$

therefore w is Hölder continuous in $\overline{x}$ according to what we have proved in part 1). As for function v, it belongs to $V_{\overline{r}}$, therefore we can apply to it lemma 3, obtaining the existence of a constant K_7, depending on n, Γ and on the coefficients of the bilinear form $a(.,.)$, such that for any r with $0 < r \leq \overline{r}$ it turns out

$$\begin{aligned} &||v||_{L^\infty(\Omega(\overline{x},r))} \leq \\ &\leq K_7 \left[||f_o||_{L^{np/(n+p)}(\Omega(\overline{x},\overline{r}))} + \sum_{i=1}^n ||f_i||_{L^p(\Omega(\overline{x},\overline{r}))} + ||h||_{L^{\overline{p}}(\Gamma(\overline{x},\overline{r}))} \right] r^{1-n/p} \end{aligned} \tag{58}$$

From the preceding arguments we arrive at the conclusion by proceeding for example as in [18].

3) Finally, it remains to consider the general case, when also the coefficients $g,\ c,\ d_i$ $(i = 1,2,\ldots,n)$ of the bilinear form $a(.,.)$ may be different from zero. To this end we can proceed once more as in [18]. Because of theorem 2, the solution u of the equaztion (48) is essentially bounded in a neighborhood U of $\overline{x}$, so that we can rewrite (48) in the form

$$\begin{aligned} &\int_\Omega \{\sum_{i,j=1}^n a_{ij}u_{x_i}v_{x_j} + \sum_{i=1}^n b_i u_{x_i} v\}\,dx = \\ &= \int_\Omega (f_o v + \sum_{i=1}^n f_i v_{x_i} - \sum_{i=1}^n d_i u v_{x_i} - cuv)\,dx + \int_\Gamma (hv - guv)\,d\sigma \\ &\qquad \forall v \in H^1(\Omega\cap U),\ v = 0 \text{ on } (\partial\Omega \cap U)\backslash\Gamma \end{aligned} \tag{59}$$

Formula (59) is an equation of the same kind of (48), but the coefficients d_i $(i = 1, 2 \ldots, n)$ and the functions c, g in it are zero. Furthermore, according to what we have already remarked, u is essentially bounded in U, in such a way that $d_i u \in L^p(U)$, $c \in L^{p/2}(U)$, $gu \in L^{\overline{p}}(U \cap \Gamma)$. The situation is now the same of case 2), so the conclusion follows. □

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