Bollettino U. M. I.
(7) 11-B (1997), 531-538

A Maximum Principle for Mixed Boundary Value Problems for Elliptic Equations in Non-Divergence Form.

MAURIZIO CHICCO (*)

Sunto. – *Si prova un principio di massimo per soluzioni di problemi al contorno misti, per equazioni differenziali lineari ellittiche del secondo ordine in forma non variazionale e a coefficienti discontinui.*

1. – Introduction.

We consider a linear second order uniformly elliptic partial differential equation in non-divergence form:

$$Lu := -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = f \tag{1}$$

in an open bounded subset Ω of $\mathbb{R}^n$. Recall the Alexandrov-Bakel'man-Pucci maximum principle for the Dirichlet problem: if $u \in H^{2,n}(\Omega)$ is such that $Lu \leq f$ a.e. in Ω with $u \leq 0$ on $\partial\Omega$ and $c \geq 0$, then

$$u \leq K\|f\|_{L^n(\Omega)} \quad \text{a.e. in } \Omega \tag{2}$$

where K is a constant depending on n, Ω and the coefficients of L (for an estimate of the constant K see e.g. [1]). This result for the Dirichlet problem has been extended later by Lions, Trudinger and Ur-

(*) I am pleased to thank Prof. Ugo Bruzzo who suggested a better wording of Lemma 1, Prof. Paolo Manselli for his bibliographic help, Dr. Laura Servidei for correcting English style, Prof. Giorgio Talenti for his useful remarks. A particular thank to the referee for his several valuable improvements.

bas [4] and Lieberman [3] to the third boundary value problem

$$\begin{cases} Lu=f \quad \text{a.e. in } \Omega, \\ Mu:=\sum_{i=1}^{n}\beta_i \dfrac{\partial u}{\partial x_i}+\gamma u=0 \quad \text{on } \partial\Omega, \end{cases}$$

or, more generally, to the mixed problem

$$\begin{cases} Lu=f \quad \text{in } \Omega, \\ Mu=0 \quad \text{on } \Gamma_1\subset\partial\Omega, \\ u=0 \quad \text{on } \partial\Omega\backslash\Gamma_1 \end{cases}$$

(for this terminology, see e.g. [5]).

More precisely, in [3] and [4] the authors assume

$$Mu\leqslant 0 \tag{3}$$

on a subset Γ_1 of $\partial\Omega$ and

$$u\leqslant 0 \tag{4}$$

on $\Gamma_0:=\partial\Omega\backslash\Gamma_1$. In [4] the set Ω is assumed convex, while in [3] the function γ is assumed strictly positive (thus excluding e.g. the Neumann problem $\partial u/\partial N=0$ on Γ_1).

The aim of the present note is to prove an inequality similar to (2) under the boundary conditions (3), (4) and $\gamma\geqslant 0$ on Γ_1. The proof is independent on those of [3] and [4], although some more regularity on β, Γ_1 and b_i is assumed.

2. - Notations and hypotheses.

Let Ω be an open bounded subset of $\mathbb{R}^n$; Γ_0, Γ_1 subsets of $\partial\Omega$, Γ_0 closed, $\Gamma_1=\partial\Omega\backslash\Gamma_0$, $\overline{\Gamma_1}$ locally of class C^2. Let β be a vector valued function of class C^2 in a neighborhood of $\overline{\Gamma_1}$, such that $|\beta|\equiv 1$ and $\beta\cdot N>0$ on $\overline{\Gamma_1}$ (where N denotes the outer normal to $\partial\Omega$). Let γ be a non-negative function defined on Γ_1. The coefficients of L (a_{ij}, b_i, c) are supposed to belong to $L^\infty(\Omega)$, and there exist two positive constants m_0, c_0 such that

$$\sum_{i,j=1}^{n} a_{ij}t_i t_j\geqslant m_0|t|^2,\ \forall t\in\mathbb{R}^n \text{ a.e. in } \Omega, \qquad c\geqslant c_0 \text{ a.e. in } \Omega. \tag{5}$$

If $u \in H^{2,n}(\Omega)$ we shall say that

$$Mu := \sum_{i=1}^{n} \beta_i \frac{\partial u}{\partial x_i} + \gamma u \leq 0 \text{ on } \Gamma_1, \quad u \leq 0 \text{ on } \Gamma_0$$

in the sense of $H^{2,n}(\Omega)$ if there exists a sequence $\{u_k\} \subset C^2(\overline{\Omega})$ such that $Mu_k \leq 0$ on Γ_1, $u_k \leq 0$ on $\Gamma_0 \, \forall k$ and $\lim_k u_k = u$ in $H^{2,n}(\Omega)$ (see e.g. Stampacchia [7]).

3. – Lemmata.

LEMMA 1. – *Let $x_0 \in \overline{\Gamma_1}$. Then there exist a positive number r and a function α, defined in a neighborhood of x_0, such that:*

$\alpha \in C^2$, $\alpha(x) \equiv 1$ if $|x - x_0| \leq r$, $0 \leq \alpha(x) \leq 1$ $\forall x$, $\alpha(x) \equiv 0$ if $|x - x_0| \geq 2r$,

$|\nabla \alpha| \leq K_1/r$ (where K_1 is a constant depending on n, β, Γ_1), $\partial \alpha / \partial \beta \equiv 0$ on Γ_1.

Furthermore, the number r can be chosen independent on x_0.

PROOF. – Consider, to begin with, $x_0 \in \overline{\Gamma_1}$ fixed. In a neighborhood of x_0 we may take local coordinates $(\xi_1, \xi_2, \ldots, \xi_n)$, where $(\xi_1, \xi_2, \ldots, \xi_{n-1})$ are coordinates on Γ_1 around x_0, and ξ_n is such that $\beta = \partial / \partial \xi_n$. We may assume that $\xi_n = 0$ on Γ_1, $(\xi_1, \xi_2, \ldots, \xi_{n-1})$ run over a subset U of $\mathbb{R}^{n-1}$, and that ξ_n takes values in $(-\delta, \delta)$. We denote by ϕ the change of coordinates that expresses the standard coordinates in $\mathbb{R}^n$ around x_0 as functions of the ξ's. Furthermore, the neighborhood U and the number δ may be chosen in such a way that ϕ is invertible.

Now, let $0 < \varrho_1 < \varrho_2$ and θ be a function of class C^2 such that $\theta(t) \equiv 1$ when $0 \leq t \leq \varrho_1$, $0 \leq \theta(t) \leq 1$ when $\varrho_1 \leq t \leq \varrho_2$, $\theta(t) \equiv 0$ when $t \geq \varrho_2$, $-2/(\varrho_2 - \varrho_1) \leq \theta'(t) \leq 0$ $\forall t$. Define also the function $\tilde{\alpha}\colon \mathbb{R}^n \to \mathbb{R}$ by $\tilde{\alpha}(\xi) = \theta(|\xi|)$, $\xi \in \mathbb{R}^n$; clearly $\partial \tilde{\alpha} / \partial \xi_n = 0$ when $\xi_n = 0$.

We choose $\varrho_2 > 0$ so that the support of $\tilde{\alpha}$ is contained in $U \times [-\delta, \delta]$ and let α be defined by

$$\tilde{\alpha}(\xi) := \alpha[\phi(\xi)], \qquad \xi \in U \times [-\delta, \delta] \tag{6}$$

where ϕ is the change of variables introduced before. According to the previous choices and since ϕ is invertible, it turns out that $\alpha(x) \equiv 1$ in a neighborhood of x_0 and $\alpha(x) \equiv 0$ if $|x - x_0|$ is sufficiently large. Therefore by choosing suitably ϱ_1, ϱ_2, r it is possible to get, as requested, $\alpha(x) \equiv 1$ if $|x - x_0| \leq r$, $\alpha(x) \equiv 0$ if $|x - x_0| \geq 2r$,

$|\nabla \alpha(x)| \leqslant K_2/r$, where K_2 is a constant depending only on ϕ, i. e. on β and Γ_1.

Moreover, we have

$$\frac{\partial \alpha}{\partial \beta} = \frac{\partial \tilde{\alpha}}{\partial \xi_n}$$

which vanishes on Γ_1.

Take now x instead of x_0 and let it vary in $\overline{\Gamma_1}$. Call $r(x)$ the supremum of the numbers r, corresponding to x, whose existence was proved before. It is easy to ascertain that the function $x \to r(x)$ is continuous on $\overline{\Gamma_1}$, therefore one can take $r := \min\{r(x)\colon x \in \overline{\Gamma_1}\} > 0$, which does not depend on x. ■

REMARK. – The above introduced function θ is of class C^2 and $\theta(t) \equiv 1$ when $0 \leqslant t \leqslant \varrho_1$, therefore $\lim_{t \to \varrho_1} \theta'(t) = \lim_{t \to \varrho_1} \theta''(t) = 0$. Since θ is supposed to be decreasing, one can claim that for any $\varepsilon > 0$ there exists $t_\varepsilon \in (0,1)$ such that $|\theta'(t)| + |\theta''(t)| < \varepsilon$ if $\theta(t) > t_\varepsilon$. By the definition of the function α, a similar property is true for it also: for any $\varepsilon > 0$ there exists $t_\varepsilon \in (0,1)$ such that $|\nabla \alpha(x)| + |D^2 \alpha(x)| < \varepsilon$ whenever $\alpha(x) > t_\varepsilon$.

LEMMA 2. – *Assumptions: let* $u \in C^2(\Omega) \cap C^1(\overline{\Omega}), f \in L^n(\Omega)$, $Lu \leqslant f$ *a.e. in* Ω, $u \leqslant 0$ *on* Γ_0, $Mu \leqslant 0$ *on* Γ_1; *there exist a vector* $\beta_0 \in \mathbb{R}^n$ *with* $|\beta_0| = 1$, *and two numbers* γ_0, δ *such that* $\gamma_0 > 0$, $0 < \delta < 1$ *and, when* $x \in \Gamma_1$ *with* $\gamma(x) \leqslant \gamma_0$, *it turns out* $|\beta(x) - \beta_0| \leqslant \delta$.

Conclusion: there exists a constant K_3, *depending on* n, m_0, γ_0, δ, $\sum \|b_i\|_{L^n(\Omega)}$, $\operatorname{diam}(\Omega)$ *such that*

$$u(x) \leqslant K_3 \|f\|_{L^n(\Omega)}, \qquad \forall x \in \Omega . \tag{7}$$

PROOF. – Replacing u with $\max(u, 0)$ and Ω with $\{x \in \Omega\colon u(x) > 0\}$, we may assume $u \geqslant 0$ in Ω. After this change, the set Γ_0 may become bigger and Γ_1 smaller, but (as one can easily verify) the condition $Mu \leqslant 0$ continues to hold in the new Γ_1.

From the hypotheses there exists in $\overline{\Omega}$ at least a maximum point for u: we may suppose that it is the origin O of $\mathbb{R}^n$, i.e.

$$u(x) \leqslant u(O) := \overline{m} \qquad \forall x \in \Omega . \tag{8}$$

Given a (fixed) $y \in \mathbb{R}^n$, let us consider the function $g(x) := u(x) - xy$, and let $\overline{x} \in \overline{\Omega}$ be such that $g(x) \leqslant g(\overline{x})\ \forall x \in \overline{\Omega}$. Let us suppose, at first, that $\bar{x} \in \Gamma_0$. Then

$$g(O) = \bar{m} \leqslant u(\bar{x}) - \bar{x}y \leqslant -\bar{x}y \leqslant |y|d \tag{9}$$

where d is the diameter of Ω.

Now consider the case $\bar{x} \in \Gamma_1$ and $\gamma(\bar{x}) > \gamma_0$. Since $(\partial g / \partial \beta)(\bar{x}) \geqslant 0$ and $Mu(\bar{x}) \leqslant 0$ by hypothesis, we get

$$\gamma_0 u(\bar{x}) \leqslant \gamma(\bar{x}) u(\bar{x}) \leqslant -\frac{\partial u}{\partial \beta}(\bar{x}) \leqslant -\beta y \leqslant |y|$$

whence

$$g(O) = \bar{m} \leqslant g(\bar{x}) = u(\bar{x}) - \bar{x}y \leqslant \frac{|y|}{\gamma_0} - \bar{x}y \leqslant \left(\frac{1}{\gamma_0} + d\right)|y| \,. \tag{10}$$

Finally, let us suppose $\bar{x} \in \Gamma_1$ with $0 \leqslant \gamma(\bar{x}) \leqslant \gamma_0$. Then by our hypotheses there exist $\beta_0 \in \mathbb{R}^n$ with $|\beta_0| = 1$ and δ with $0 < \delta < 1$ such that $|\beta(\bar{x}) - \beta_0| \leqslant \delta$. Since we have

$$Mu(\bar{x}) = \frac{\partial u}{\partial \beta}(\bar{x}) + \gamma(\bar{x}) u(\bar{x}) \leqslant 0,$$

$$\frac{\partial g}{\partial \beta}(\bar{x}) = \frac{\partial u}{\partial \beta}(\bar{x}) - \beta(\bar{x}) y \geqslant 0\,,$$

we get $\beta(\bar{x}) y \leqslant 0$, whence

$$\beta_0 y \leqslant \delta |y| \,. \tag{11}$$

Therefore if we choose $y \in \mathbb{R}^n$ such that neither (9) nor (10) nor (11) is satisfied, i.e.

$$|y| < \frac{\bar{m}\gamma_0}{1 + d\gamma_0}, \qquad \beta_0 y > \delta |y| \tag{12}$$

the point $\bar{x}$ will belong neither to Γ_0 nor to Γ_1, hence is necessarily interior to Ω. In this case obviously $\nabla g(\bar{x}) = 0$ i.e. $\nabla u(\bar{x}) = y$; this means that the set

$$T = \left\{ y \in \mathbb{R}^n \colon |y| < \frac{\bar{m}\gamma_0}{1 + d\gamma_0}, \beta_0 y > \delta |y| \right\}$$

is contained in $\nabla u(\Gamma_u)$, which denotes the set of the values assumed in Ω by the vector function $x \to \nabla u(x)$ (see [6], [2]).

Therefore the result may be proved as in [2] Theorem 9.1, with the only difference that in Lemma 9.4 the integration will be extended to the above defined set T instead of the whole ball. So the proof is complete, with the constant K_3 obtained as in [2] but depending also on γ_0 and δ as explained. ■

THEOREM. – *Assume the hypotheses listed in section 2. Suppose* $u \in H^{2,n}(\Omega)$ *such that* $Mu \leq 0$ *on* Γ_1, $u \leq 0$ *on* Γ_0 *in the sense of* $H^{2,n}(\Omega)$, $Lu \leq f$ *a.e. in* Ω. *Then there exists a constant* K_4 *depending on the coefficients of L and* β, γ, Γ_1 *such that*

$$u \leq K_4 \|f\|_{L^n(\Omega)} \quad in\ \Omega. \tag{13}$$

PROOF. – Since it is possible to approximate u in the norm of $H^{2,n}(\Omega)$ by a sequence u_k ($k \in \mathbb{N}$) of functions belonging to $C^2(\overline{\Omega})$ and satisfying the same boundary conditions of u, taking into account that $Lu_k \to f$ in $L_n(\Omega)$, we may assume without loss of generality $u \in C^2(\overline{\Omega})$.

From the compactness of $\overline{\Gamma_1}$, the continuity of β and lemmata 1 and 2, the following is true: there exists a positive number r such that for any $x_0 \in \overline{\Gamma_1}$ there exists both the function α as described by Lemma 1 (with support in $B(x_0, 2r)$) and the number β_0 of Lemma 2, as applied to $\Omega \cap B(x_0, 2r)$ instead of Ω. Also, it is clear that in lemma 2 the numbers $\gamma_0 > 0$ and $\delta \in (0,1)$ can be chosen arbitrarily (for example $\gamma_0 = 1$ and $\delta = 1/2$).

Let x_M be a point of $\overline{\Omega}$ such that $u(x_M) = \max\{u(x): x \in \overline{\Omega}\} := +\overline{m}$. Suppose first that $\mathrm{dist}(x_M, \overline{\Gamma_1}) < r$; then there exists at least a point $x_0 \in \overline{\Gamma_1}$ such that $x_M \in B(x_0, r)$. Because of previous considerations we may apply Lemma 1 to the ball $B(x_0, 2r)$. Let α the function described there and define the operator L' by

$$L' := -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \left(b_i + \frac{2}{\alpha} \sum_{j=1}^{n} a_{ij} \frac{\partial \alpha}{\partial x_j} \right) \frac{\partial}{\partial x_i}. \tag{14}$$

By hypothesis, $u \leq 0$ on Γ_0 and $\partial u / \partial \beta + \gamma u \leq 0$ on Γ_1. By the definition of α we have $\partial \alpha / \partial \beta = 0$ on Γ_1, therefore

$$\frac{\partial(\alpha u)}{\partial \beta} + \gamma \alpha u = \alpha \left(\frac{\partial u}{\partial \beta} + \gamma u \right) \leq 0 \ \text{ on } \Gamma_1, \qquad \alpha u \leq 0 \ \text{ on } \Gamma_0.$$

Let $0 < t < 1$; consider the function $w := \max(\alpha u - t\overline{m}, 0)$ and observe that $w(x) = 0$ at any point x where $\alpha(x) \leq t$. Moreover in the set

$$\Omega_t := \{x \in B(x_0, 2r) \cap \Omega\colon \alpha(x)u(x) > t\overline{m}\}$$

(which is not empty since it contains x_M) we have

(15) $L'w = L'(\alpha u) =$

$$= \alpha L u + \left(- \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 \alpha}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial \alpha}{\partial x_i} + \frac{2}{\alpha} \sum_{i,j=1}^{n} a_{ij} \frac{\partial \alpha}{\partial x_i} \frac{\partial \alpha}{\partial x_j} - c\alpha \right) u .$$

An easy inspection shows that the function $w = \max(\alpha u - t\overline{m}, 0)$ satisfies the same boundary conditions as u even on the new Γ_1 and Γ_0 (after replacing Ω with Ω_t).

We need now to get rid of the last term in (15), and to this end we adjust the parameter t. By the remark after Lemma 1, we can choose $t < 1$ such that

$$(16) \quad - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 \alpha}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial \alpha}{\partial x_i} + \frac{2}{\alpha} \sum_{i,j=1}^{n} a_{ij} \frac{\partial \alpha}{\partial x_i} \frac{\partial \alpha}{\partial x_j} - c\alpha \leq 0$$

if $\alpha(x) > t$ (in fact we have supposed $c \geq c_0$, c_0 a positive constant, and $b_i \in L^\infty$). From (15), (16) we deduce $L'w \leq \alpha L u \leq \alpha f$ a. e. in Ω_t and by applying Lemma 2 we get

$$(17) \qquad w \leq \tilde{K} K \|f\|_{L^n(\Omega)} \quad \text{in } \Omega_t$$

whence, by the definition of w

$$(18) \qquad \overline{m} \leq \frac{\tilde{K}}{1-t} \|f\|_{L^n(\Omega)} .$$

The constant $\tilde{K}$ depends on the coefficients of L, on α (and therefore on Γ_1) and on β; the number t depends on α, c_0 and $\sum \|b_i\|_{L^\infty}$.

We still have to consider the case $\operatorname{dist}(x_M, \overline{\Gamma_1}) \geq r$. In this case (without applying the preceding lemmata), let α be a function such that: $\alpha \in C_0^2(\mathbb{R}^n)$, $\alpha(x) = 0$ if $|x - x_M| \geq r$, $\alpha(x) = 1$ in a neighborhood of x_M, $0 \leq \alpha(x) \leq 1$, $|\nabla \alpha(x)| \leq 2/r$. Since clearly the function αu satisfies the Dirichlet condition $\alpha u \leq 0$ on $\partial\Omega$, we may repeat the preceding procedure from (15) to (18) by applying, instead of Lemma 2, the inequality of Alexandrov-Bakel'man-Pucci (see e.g. [2] Theorem 9.1). In this way we get an inequality like (18) and we conclude as before. ■

REFERENCES

[1] X. Cabre, *On the Alexandroff-Bakel'man-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math., **48** 1995, 539-570.

[2] D. Gilbarg - N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[3] G. M. Lieberman, *Local estimates for subsolutions and supersolutions of oblique derivative problems for general second order elliptic equations*, Trans. Amer. Math. Soc., **304** (1987), 343-353.

[4] P. L. Lions - N. S. Trudinger - J. I. E. Urbas, *The Neumann problem for equations of Monge-Ampère type*, Comm. Pure Appl. Math., **39** (1986), 539-563

[5] C. Miranda, *Partial differential equations of elliptic type*, 2nd ed., Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[6] C. Pucci, *Limitazioni per soluzioni di equazioni ellittiche*, Ann. Mat. Pura Appl. (4), **74** (1966), 15-30.

[7] G. Stampacchia, *Équations elliptiques du seconde ordre à coefficients discontinus*, Université de Montreal, 1966.

Istituto Matematico di Ingegneria
Università di Genova, P.le Kennedy, Pad. D - 16129 Genova
e-mail address: chicco@dima.unige.it

Pervenuta in Redazione
il 20 aprile 1995 e, in forma rivista, il 10 gennaio 1997