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Solvability of the Dirichlet Problem in $H^{2,p}(\Omega)$ for a Class of Linear Second Order Elliptic Partial Differential Equations.

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Summary. – *I study the Dirichlet problem $Lu = f$ in Ω, $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$ with given $f \in L_p(\Omega)$, $1 < p < +\infty$. Here L is a linear second order uniformly elliptic partial differential operator, where the coefficients of the second derivatives are (uniformly) continuous in Ω, while the other ones belong to suitable $L_q(\Omega)$ classes.*

1. – Introduction.

We consider the elliptic operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c \tag{1}$$

where the coefficients a_{ij} are uniformly continuous in the open set Ω and the other ones belong to suitable $L_q(\Omega)$ spaces. Many authors (see for example [8], [9], [11]) have studied the inequality

$$\|u\|_{H^{2,p}(\Omega)} \leqslant K_1 \{ \|Lu\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)} \} , \tag{2}$$

with $1 < p < +\infty$, valid for any function u which vanishes on the boundary of Ω and possesses generalized second derivatives in $L_p(\Omega)$. The constant K_1 depends on p, n, Ω and the coefficients of L. The aim of the present work is to study, starting from (2), the solvability of the following Dirichlet problem: given any $f \in L_p(\Omega)$ (with $1 < p < +\infty$), to establish existence and uniqueness of a

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function u such that

$$(3)\qquad \begin{cases} Lu = f & \text{a.e. in } \Omega, \\ u \in H^{2,p}(\Omega)\,, \ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The principal result claims that if the essential infimum of c in Ω is positive, the problem (3) has one and only one solution (see theorem 2).

The particular case $p = 2$ was the subject of my earlier work [4]. I wish to thank G. TALENTI, to whom I owe the suggestion of extending those results to the general case $1 < p < +\infty$.

2. – Notations and hypotheses.

Let Ω be an open bounded set in R^n, with $n \geqslant 2$.

We suppose that the boundary of Ω (denoted by $\partial\Omega$) can be represented locally by a function with continuous second derivatives. Let us put, for shortness:

$$\|u_x\|_{L_p(\Omega)} = \sum_{i=1}^{n} \|u_{x_i}\|_{L_p(\Omega)}\,; \quad \|u_{xx}\|_{L_p(\Omega)} = \sum_{i,j=1}^{n} \|u_{x_i x_j}\|_{L_p(\Omega)}\,.$$

We denote by $H^{1,p}(\Omega)$, $\overset{0}{H}{}^{1,p}(\Omega)$ the Banach spaces obtained by completing $C^1(\bar\Omega)$ and $C^1_0(\Omega)$ respectively according to the norm

$$\|u\|_{H^{1,p}(\Omega)} = \|u\|_{L_p(\Omega)} + \|u_x\|_{L_p(\Omega)}\,.$$

Let $H^{2,p}(\Omega)$ denote the space obtained by completing $C^2(\bar\Omega)$ according to the norm

$$(4)\qquad \|u\|_{H^{2,p}(\Omega)} = \|u\|_{L_p(\Omega)} + \|u_x\|_{L_p(\Omega)} + \|u_{xx}\|_{L_p(\Omega)}\,.$$

We observe that in $H^{2,p}(\Omega) \cap \overset{0}{H}{}^{1,p}(\Omega)$ the norm (4) is equivalent to $\|u_{xx}\|_{L_p(\Omega)}$: see [7]. This fact will be often used throughout the present work without mention.

Let L be the operator defined in (1); we suppose that $a_{ij} = a_{ji}$,

$$a_{ij} \in C^0(\bar\Omega)\,, \quad b_i \in L_r(\Omega)\,, \quad c \in L_s(\Omega)\,, \quad (i, j = 1, 2, \ldots, n)$$

where $r = n$ for $1 < p < n$, $r > n$ for $p = n$, $r = p$ for $p > n$; $s = n/2$ for $1 < p < n/2$, $s > n/2$ for $p = n/2$, $s = p$ for $p > n/2$. There exi -sts a positive constant ν such that $\sum_{i,j=1}^{n} a_{ij} t_i t_j > \nu |t|^2$ in $\bar\Omega$.

3. – Preliminary lemmas.

We set

$$\tilde{L} = -\sum_{i,j=1}^{n} \tilde{a}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \tilde{b}_i \frac{\partial}{\partial x_i} + \tilde{c} \tag{5}$$

where $\tilde{a}_{ij}$, $\tilde{b}_i$, $(i, j = 1, 2, \dots, n)$, $\tilde{c} \in C^2(\bar{\Omega})$ and

$$\sum_{i,j=1}^{n} \tilde{a}_{ij} t_i t_j \geqslant \nu |t|^2 \text{ in } \bar{\Omega}, \tag{6}$$

$$\max_{\bar{\Omega}} \sum_{i,j=1}^{n} \left| \frac{\partial \tilde{a}_{ij}}{\partial x_j} \right| = K_2, \tag{7}$$

$$\max_{\bar{\Omega}} \left(\sum_{i=1}^{n} |\tilde{b}_i| + |\tilde{c}| \right) = K_3. \tag{8}$$

LEMMA 1. – *There exists a positive constant λ_0, depending on ν, K_2, K_3, p, n, such that*

$$\|u\|_{L_p(\Omega)} \leqslant (\lambda - \lambda_0)^{-1} \|\tilde{L}u + \lambda u\|_{L_p(\Omega)}$$

for any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$ and any $\lambda > \lambda_0$.

PROOF. – We begin with supposing $p \geqslant 2$; the remaining case $1 < p < 2$ will be discussed later (see page 12). For any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$ we have:

$$\begin{aligned}\int_\Omega (\tilde{L}u + \lambda u)|u|^{p-1} \operatorname{sign} u \, dx &= \int_\Omega \Big\{ (p-1)|u|^{p-2} \sum_{i,j=1}^{n} \tilde{a}_{ij} u_{x_i} u_{x_j} + \\ &+ \sum_{i=1}^{n} \Big[\sum_{j=1}^{n} (\tilde{a}_{ij})_{x_j} + \tilde{b}_i \Big] u_{x_i} |u|^{p-1} \operatorname{sign} u + (\lambda + \tilde{c})|u|^p \Big\} dx \geqslant \\ &\geqslant \int_\Omega \Big\{ \nu(p-1)|u|^{p-2} \sum_{i=1}^{n} u_{x_i}^2 + \sum_{i=1}^{n} \Big[\sum_{j=1}^{n} (\tilde{a}_{ij})_{x_j} + \tilde{b}_i \Big] \cdot \\ &\qquad \cdot u_{x_i} |u|^{p-1} \operatorname{sign} u + (\lambda + \tilde{c})|u|^p \Big\} dx;\end{aligned} \tag{9}$$

$$(10)\qquad \left|\int_\Omega \sum_{i=1}^n \left[\sum_{j=1}^n (\tilde a_{ij})_{x_j} + \tilde b_i\right] u_{x_i}|u|^{p-1}\operatorname{sign} u\, dx\right| \le$$

$$\le (K_2+K_3)\int_\Omega |u|^{p-1}\sum_{i=1}^n |u_{x_i}|\,dx \le n(K_2+K_3)\left(\int_\Omega |u|^{p-2}\sum_{i=1}^n u_{x_i}^2\,dx\right)^{\frac12}\cdot$$

$$\cdot\left(\int_\Omega |u|^p\,dx\right)^{\frac12} \le n(K_2+K_3)\left(\eta\int_\Omega |u|^{p-2}\sum_{i=1}^n u_{x_i}^2\,dx + \frac{1}{4\eta}\int_\Omega |u|^p\,dx\right)$$

where η is any positive number.

$$(11)\qquad \left|\int_\Omega \tilde c|u|^p\,dx\right| \le K_3\int_\Omega |u|^p\,dx\,.$$

Let us choose now η and λ_0 in the following way:

$$\eta = \frac{\nu(p-1)}{2n(K_2+K_3)}\,; \qquad \lambda_0 = K_3 + \frac{n^2(K_2+K_3)^2}{2\nu(p-1)}\,.$$

Then from (9), (10), (11) and for $\lambda > \lambda_0$ we have

$$(12)\qquad \int_\Omega (\tilde L u + \lambda u)|u|^{p-1}\operatorname{sign} u\,dx \ge$$

$$\ge \frac{\nu(p-1)}{2}\int_\Omega |u|^{p-2}\sum_{i=1}^n u_{x_i}^2\,dx + (\lambda-\lambda_0)\int_\Omega |u|^p\,dx\,.$$

A simple use of Hölder's inequality in (12) concludes the proof when $p\ge 2$. ■

LEMMA 2. – *For any $\varepsilon > 0$ there exists an operator $\tilde L$ of the type* (5) *such that it results*

$$\|\tilde L u - Lu\|_{L_p(\Omega)} \le \varepsilon\|u_{xx}\|_{L_p(\Omega)} \qquad \forall u \in H^{2,p}(\Omega)\cap H_0^{1,p}(\Omega)\,.$$

PROOF. – For simplicity let us confine ourselves to the case $1 < p < n/2$, the other cases being similar.

We have:

$$(13)\qquad \|\tilde L u - Lu\|_{L_p(\Omega)} \le n^{2(p-1)/p}\max_{\bar\Omega}\sum_{i,j=1}^n |\tilde a_{ij} - a_{ij}|\,\|u_{xx}\|_{L_p(\Omega)} +$$

$$+ n^{(p-1)/p}\sum_{i=1}^n \|\tilde b_i - b_i\|_{L_n(\Omega)}\|u_x\|_{L_{pn/(n-p)}(\Omega)} + \|\tilde c - c\|_{L_{n/2}(\Omega)}\|u\|_{L_{pn/(n-2p)}(\Omega)}\,.$$

For known results (see for example [7]) there exist positive constants K_4, K_5 depending on p, n, Ω such that

$$\|u_x\|_{L_{pn/(n-p)}(\Omega)} \leqslant K_4 \|u_{xx}\|_{L_p(\Omega)} , \tag{14}$$

$$\|u\|_{L_{pn/(n-2p)}(\Omega)} \leqslant K_5 \|u_{xx}\|_{L_p(\Omega)} \tag{15}$$

for any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$. Since $C^2(\bar{\Omega})$ is dense in $C_0(\bar{\Omega})$ and in $L_q(\Omega)$ $(1 < q < +\infty)$, (13), (14), (15) yield the assertion. ■

LEMMA 3. – *For any $\varepsilon > 0$ there exist constants K_6, K_7 depending on ε, p, n, Ω, b_i, c $(i = 1, 2, \dots, n)$ such that*

$$\sum_{i=1}^{n} \|b_i u_{x_i}\|_{L_p(\Omega)} \leqslant \varepsilon \|u_{xx}\|_{L_p(\Omega)} + K_6 \|u\|_{L_p(\Omega)} , \tag{16}$$

$$\|cu\|_{L_x(\Omega)} \leqslant \varepsilon \|u_{xx}\|_{L_p(\Omega)} + K_7 \|u\|_{L_p(\Omega)} \tag{17}$$

for any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$.

PROOF. – Let us confine ourselves, for simplicity, to the case $1 < p < n/2$. For any $\eta > 0$ we can write

$$b_i = b_i' + b_i'' \qquad (i = 1, 2, \dots, n) ,$$
$$c = c' + c''$$

in such a way that

$$\sum_{i=1}^{n} \|b_i'\|_{L_n(\Omega)} \leqslant \eta , \qquad \|c'\|_{L_{n/2}(\Omega)} \leqslant \eta ,$$
$$\operatorname*{ess\,sup}_{\Omega} \left(|c''| + \sum_{i=1}^{n} |b_i''| \right) = K_8 < +\infty .$$

Whence it follows

$$\sum_{i=1}^{n} \|b_i u_{x_i}\|_{L_p(\Omega)} \leqslant \eta \sum_{i=1}^{n} \|u_{x_i}\|_{L_{pn/(n-p)}(\Omega)} + K_8 \|u_x\|_{L_p(\Omega)} , \tag{18}$$

$$\|cu\|_{L_p(\Omega)} \leqslant \eta \|u\|_{L_{pn/(n-2p)}(\Omega)} + K_8 \|u\|_{L_p(\Omega)} . \tag{19}$$

From (14), (15), (18), (19) we get

$$\sum_{i=1}^{n} \|b_i u_{x_i}\|_{L_p(\Omega)} \leqslant \eta K_4 \|u_{xx}\|_{L_p(\Omega)} + K_8 \|u_x\|_{L_p(\Omega)} , \tag{20}$$

$$\|cu\|_{L_p(\Omega)} \leqslant \eta K_5 \|u_{xx}\|_{L_p(\Omega)} + K_8 \|u\|_{L_p(\Omega)} . \tag{21}$$

We use now the inequality (see e.g. [7] page 122):

$$\|u_x\|_{L_p(\Omega)} \leqslant \eta \|u_{xx}\|_{L_p(\Omega)} + K_9 \|u\|_{L_p(\Omega)} \tag{22}$$

valid for any $\eta > 0$ and any $u \in H^{2,p}(\Omega)$, where K_9 depends on η, p, n, Ω. From (20), (21), (22) it is easy to obtain (16), (17). ∎

LEMMA 4. – *There exists a constant K_{10} depending on p, n, Ω and the coefficients in L such that*

$$\|u_{xx}\|_{L_p(\Omega)} \leqslant K_{10}\{\|Lu\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)}\} \tag{23}$$

for any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$.

PROOF. – In the articles [8], [9] (see also [11] page 193) the following inequality is proven:

$$\|u_{xx}\|_{L_p(\Omega)} \leqslant K_{11}\left\{\left\|\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j}\right\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)}\right\} \tag{24}$$

valid for any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$. The constant K_{11} depends on p, n, ν, Ω and the modulus of continuity of the coefficients a_{ij}. From lemma 3 it follows, for any $\varepsilon > 0$:

$$\left\|\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j}\right\|_{L_p(\Omega)} \leqslant \|Lu\|_{L_p(\Omega)} + \sum_{i=1}^{n} \|b_i u_{x_i}\|_{L_p(\Omega)} + \\ + \|cu\|_{L_p(\Omega)} \leqslant \|Lu\|_{L_p(\Omega)} + 2\varepsilon \|u_{xx}\|_{L_p(\Omega)} + (K_6 + K_7)\|u\|_{L_p(\Omega)} . \tag{25}$$

From (24), (25) it is easy to reach (23). ∎

4. – Main results.

THEOREM 1. – *There exist two positive constants K_{12}, $\hat{\lambda}$, depending on n, p, Ω and the coefficients of L, such that*

$$\|u_{xx}\|_{L_p(\Omega)} \leqslant K_{12} \|Lu + \lambda u\|_{L_p(\Omega)} \tag{26}$$

for any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$ and uniformly for any $\lambda \geqslant \hat{\lambda}$.

PROOF. – Starting from (23) we get easily

$$\|u_{xx}\|_{L_p(\Omega)} \leqslant K_{10}\{\|Lu + \lambda u\|_{L_p(\Omega)} + (\lambda + 1)\|u\|_{L_p(\Omega)}\} . \tag{27}$$

Let $\tilde{L}$ be an operator like (5) such that

$$\|\tilde{L}u - Lu\|_{L_p(\Omega)} \leqslant (2K_{10})^{-1}\|u_{xx}\|_{L_p(\Omega)} \tag{28}$$

for any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$. The existence of such an $\tilde{L}$ is guaranteed by lemma 2. Let λ_0 be the constant defined in lemma 1 corresponding to this $\tilde{L}$, so that

$$\|u\|_{L_p(\Omega)} \leqslant (\lambda - \lambda_0)^{-1}\|\tilde{L}u + \lambda u\|_{L_p(\Omega)} \tag{29}$$

for any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$ and any $\lambda > \lambda_0$.

Using (27), (28), (29) we find

$$\begin{aligned}\|u_{xx}\|_{L_p(\Omega)} &\leqslant K_{10}[\|Lu + \lambda u\|_{L_p(\Omega)} + (\lambda + 1)(\lambda - \lambda_0)^{-1}\cdot \\ &\cdot\|\tilde{L}u + \lambda u\|_{L_p(\Omega)}] \leqslant K_{10}[1 + (\lambda + 1)(\lambda - \lambda_0)^{-1}]\|Lu + \lambda u\|_{L_p(\Omega)} + \\ &+ (\lambda + 1)2^{-1}(\lambda - \lambda_0)^{-1}\|u_{xx}\|_{L_p(\Omega)}\,.\end{aligned} \tag{30}$$

Choose now $\hat{\lambda} = 2 + 3\lambda_0$: from (30) it is easy to get

$$\|u_{xx}\|_{L_p(\Omega)} \leqslant 10K_{10}\|Lu + \lambda u\|_{L_p(\Omega)} \tag{31}$$

for any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$ and uniformly for any $\lambda \geqslant \hat{\lambda}$. ■

COROLLARY 1. – *We suppose that $\lambda \geqslant \hat{\lambda}$, where $\hat{\lambda}$ is the constant introduced in theorem* 1; *let f be given in $L_p(\Omega)$. Then the Dirichlet problem*

$$\begin{cases} Lu + \lambda u = f & \text{a.e. in } \Omega \\ u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega) \end{cases} \tag{32}$$

has one and only one solutiono Moreover, if we suppose $c \geqslant 0$ a.e. in Ω, $f \geqslant 0$ a.e. in Ω it follows $u \geqslant 0$ a.e. in Ω.

PROOF. – Let us extend the definition of the coefficients a_{ij} to all of R^n: denoting with the same letters the extended coefficients, we suppose that $a_{ij} \in C^0(R^n)$ $(i, j = 1, 2, \ldots, n)$, $\sum_{i,j=1}^{n} a_{ij}t_i t_j \geqslant \nu|t|^2$ in R^n. Then we extend the definition of b_i, c, f to all of R^n by setting $b_i(x) \equiv c(x) \equiv f(x) \equiv 0$ in $R^n - \Omega$ $(i = 1, 2, \ldots, n)$.

Let θ be a function in $C_0^\infty(R^n)$ such that:

$$\int_{R^n}\theta(x)\,dx = 1\,, \qquad \theta \geqslant 0 \text{ in } R^n\,, \qquad \theta(x) = 0 \text{ when } |x| \geqslant 1\,.$$

We set, for $m = 1, 2, \ldots$:

$$a_{ij}^{(m)}(x) = m^{-n}\int_{R^n}\theta\left(\frac{x-y}{m}\right)a_{ij}(y)\,dy \qquad (i, j = 1, 2, \ldots, n)$$

and similarly $b_i^{(m)}, c^{(m)}, f^{(m)}$. It turns out

$$\text{(33)} \qquad \lim_{n\to+\infty}\max_{\Omega}\sum_{i,j=1}^{n}|a_{ij}-a_{ij}^{(m)}| = 0\,,$$

$$\text{(34)} \qquad \lim_{n\to+\infty}\left\{\sum_{i=1}^{n}\|b_i - b_i^{(m)}\|_{L_r(\Omega)} + \|c-c^{(m)}\|_{L_s(\Omega)} + \|f-f^{(m)}\|_{L_p(\Omega)}\right\} = 0$$

and $a_{ij}^{(m)}, b_i^{(m)}, c^{(m)}, f^{(m)} \in C^\infty(R^n)$ $(m = 1, 2, \ldots)$.

Set now

$$\text{(35)} \qquad L^{(m)} = -\sum_{i,j=1}^{n}a_{ij}^{(m)}\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{n}b_i^{(m)}\frac{\partial}{\partial x_i} + c^{(m)}.$$

By controlling the previous proofs it is easy to check that inequality (26), written for the operators $L^{(m)}$:

$$\text{(36)} \qquad \|u_{xx}\|_{L_p(\Omega)} \leqslant K_{12}\|L^{(m)}u + \lambda u\|_{L_p(\Omega)}$$

is valid for any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$ and any $\lambda \geqslant \hat{\lambda}$, with the same constants $\hat{\lambda}$ and K_{12} uniformly with respect to m. This implies the uniqueness of the solutions $u^{(m)}$ of the Dirichlet problems

$$\text{(37)} \qquad \begin{cases} L^{(m)}u^{(m)} + \lambda u^{(m)} = f^{(m)} & \text{in } \Omega, \\ u^{(m)} \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega) & \text{for } m = 1, 2, \ldots. \end{cases}$$

For known results, since the operators $L^{(m)}$ have regular coefficients, Riesz-Fredholm theory can be applied to them so that the uniqueness of the solutions of problems (37) implies their existence. From (36), (37), it follows

$$\text{(38)} \qquad \|u_{xx}^{(m)}\|_{L_p(\Omega)} \leqslant K_{12}\|f^{(m)}\|_{L_p(\Omega)} \qquad m = 1, 2, ..$$

From (38) we deduce the existence of a sequence, extracted from. $\{u^{(m)}\}_{m\in N}$, weakly converging in $H^{2,p}(\Omega)$ to a function u which is solution of problem (32): in fact, it is sufficient to pass to the limit in (37) for $m \to +\infty$ remembering (33), (34). The uniqueness of u, solution of (32), is immediate from theorem 1.

The last assertion follows from the fact that, if $c, f \geq 0$ a.e. in Ω, it is also $c^{(m)}, f^{(m)} \geq 0$ in Ω $(m = 1, 2, ...)$; whence for known theorems (see e.g. [12]) it turns out $u^{(m)} \geq 0$ in Ω $(m = 1, 2, ...)$. Since $u^{(m)}$ converges weakly to u in $H^{2,p}(\Omega)$, we get $u \geq 0$ a.e. in Ω. ∎

THEOREM 2. – *Let us suppose* $\operatorname{ess\,inf}_{\Omega} c > 0$, $f \in L_p(\Omega)$. *Then the Dirichlet problem*

$$(39) \qquad \begin{cases} Lu = f & \text{a.e. in } \Omega, \\ u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega) \end{cases}$$

has one and only one solution. If $b_i \in L_\infty(\Omega)$ *for at least one value of* i, *the conclusion is valid even if* $\operatorname{ess\,inf}_{\Omega} c = 0$. *Besides, if* $f \geq 0$ *a.e. in* Ω, *it turns out* $u \geq 0$ *a.e. in* Ω.

PROOF. – It is very similar to that of [4] and *I* give it only for completeness. Suppose $\lambda \geq \hat{\lambda}$, where $\hat{\lambda}$ is defined in theorem 1. Then, by corollary 1, there exists the inverse operator of $L + \lambda I$, denoted by G_λ, which brings $L_p(\Omega)$ onto $H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$.

Since G_λ is a compact operator in $L_p(\Omega)$, its spectrum is discrete and countable. Denoting by $\{\lambda_j\}_{j\in N}$ the sequence of the eigenvalues of L and by $\{\mu_j\}_{j\in N}$ that of the eigenvalues of G_λ, we have

$$(40) \qquad \mu_j = (\lambda + \lambda_j)^{-1} \qquad (j = 1, 2, ...)$$

for any $\lambda \geq \hat{\lambda}$. Besides, from corollary 1, we can apply theorem 6.1 of [10] to the operator G_λ since it leaves invariant the cone of non-negative functions in $L_p(\Omega)$.

Proceeding as in [2], [3], [4] we find that there exists an eigen value μ_1 of G_λ which is real and has maximum modulus among all the eigenvalues of G_λ:

$$(41) \qquad |\mu_j| \leq \mu_1 \qquad \forall \mu_j \text{ eigenvalue of } G_\lambda.$$

Let us denote by λ_1 the real eigenvalue of L corresponding to μ_1, that is

$$\lambda_1 = \frac{1 - \lambda\mu_1}{\mu_1}.$$

From (40), (41), passing to the limit for $\lambda \to +\infty$, it follows

$$(42) \qquad \operatorname{Re} \lambda_j \geq \lambda_1 \qquad \forall \lambda_j \text{ eigenvalue of } L.$$

Therefore it is clear that the assertion will be proved if we show that $\lambda_1 > 0$. To this purpose let is consider the operators $G_\lambda^{(m)}$ inverses of $L^{(m)} + \lambda I$, where $L^{(m)}$ are defined in (35). These operators $G_\lambda^{(m)}$ certainly exist if $\lambda \geqslant \hat{\lambda}$. From (33), (34) it follows

$$\lim_{m\to+\infty} \max\{\|L^{(m)}u - Lu\|_{L_p(\Omega)} : \|u_{xx}\|_{L_p(\Omega)} \leqslant 1\} = 0 . \tag{43}$$

This implies that the sequence of operators $\{G_\lambda^{(m)}\}_{m\in N}$ converges to G_λ in the uniform metric of

$$\mathfrak{L}[L_p(\Omega); H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)]$$ (see e.g. [1] lemma 3.7 or [6]).

From a lemma of [6] (page 1091) it follows

$$\lim_{m\to+\infty} \mu_j^{(m)} = \mu_j \tag{44}$$

uniformly for $j \in N$, where $\{\mu_j^{(m)}\}_{j\in N}$ is the sequence of eigenvalues of $G_\lambda^{(m)}$. In particular we have

$$\lim_{m\to+\infty} \mu_1^{(m)} = \mu_1$$

whence at once

$$\lim_{m\to+\infty} \lambda_1^{(m)} = \lambda_1 , \tag{45}$$

where $\lambda_1^{(m)}$ is the eigenvalue of $L^{(m)}$ having minimum real part, that is

$$\lambda_1^{(m)} = \frac{1 - \lambda\mu_1^{(m)}}{\mu_1^{(m)}} .$$

Let us observe now that the usual maximum principle is valid for the operators $L^{(m)}$, since they have smooth coefficients and it is $c^{(m)} \geqslant 0$ in Ω (see e.g. [12]). Moreover we have supposed $c \geqslant k$ a.e. in Ω, where k is a positive constant; it follows

$$\lambda_1^{(m)} \geqslant k , \qquad m = 1, 2, \ldots$$

From (45) we get then

$$\lambda_1 \geqslant k > 0 . \tag{46}$$

Therefore 0 is not an eigenvalue of L and problem (39) has one and only one solution.

Let us prove now that, if $b_i \in L_\infty(\Omega)$ for at least one value of i, the result is true even if $\operatorname*{ess\,inf}_\Omega c = 0$. To this end it is sufficient to use a trick by Picard just as in [5] (page 322). I refer the proof only for readers' convenience.

Let $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$, $Lu = 0$ a.e. in Ω, $\operatorname*{ess\,inf}_\Omega c = 0$, $b_1 \varepsilon L_\infty(\Omega)$: let us show that $u = 0$ a.e. in Ω. We set $u = zv$ with $z = C - \exp[hx_1]$, where C and h are constants to be determined later. We have:

$$Lu = -\sum_{i,j=1}^{n} a_{ij} z_{x_i x_j} v - \sum_{i,j=1}^{n} a_{ij} v_{x_i x_j} z - 2\sum_{i,j=1}^{n} a_{ij} z_{x_i} v_{x_j} + \\ + \sum_{i=1}^{n} b_i z_{x_i} v + \sum_{i=1}^{n} b_i v_{x_i} z + czv = 0 \text{ a.e. in } \Omega .$$

From the definition of z it follows

$$z\left[-\sum_{i,j=1}^{n} a_{ij} v_{x_i x_j} + \left(\frac{2h \exp[hx_1]}{z}\sum_{i=1}^{n} a_{i1} + \sum_{i=1}^{n} b_i\right) v_{x_i} + \\ + \left(\frac{a_{11} h^2 \exp[hx_1] - b_1 h \exp[hx_1]}{z} + c\right) v\right] = 0 \qquad \text{a.e. in } \Omega. \tag{47}$$

Now we choose the constants C and h so that $z > 1$ in Ω and

$$\operatorname*{ess\,inf}_\Omega \left(\frac{a_{11} h^2 - b_1 h}{z} \exp[hx_1] + c\right) > 0 .$$

In this way the eq. (47) becomes of the type

$$L_1 v = 0 \qquad \text{a.e. in } \Omega ,$$

where the coefficients of L_1 satisfy the hypotheses sufficient to apply the first part of the present theorem. It follows $v = 0$ a.e. in Ω, that is $u = 0$ a.e. in Ω. Therefore again 0 is not an eigenvalue of L and problem (39) has one and only one solution. It remains to prove that if $f \geqslant 0$ a.e. in Ω it follows $u \geqslant 0$ a.e. in Ω. We have already observed that the maximum principle is valid for the operator $L^{(m)}$ defined in (35), that is

$$G_0^{(m)} f \geqslant 0 \text{ in } \Omega \qquad (m = 1, 2, \ldots) \tag{48}$$

where $G_0^{(m)} = [L^{(m)}]^{-1}$. From (43) we get

$$\lim_{m\to+\infty} \|G_0 f - G_0^{(m)} f\|_{H^{2,p}(\Omega)} = 0 \tag{49}$$

where $G_0 = L^{-1}$. From (48) (49) it follows $G_0 f \geqslant 0$ a.e. in Ω, or $u \geqslant 0$ a.e. in Ω. ■

PROOF OF LEMMA 1 WHEN $1<p<2$. – Up to now lemma 1, and hence all the sequel, has been proven only for $2\leqslant p<+\infty$. The proof of lemma 1 will now be completed using a trick suggested to me by G. TALENTI.

Suppose $1<p<2$ and $\tilde{L}$ as in (5). Let us consider the operator

$$\tilde{L}^* = -\sum_{i,j=1}^{n} \tilde{a}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^* \frac{\partial}{\partial x_i} + c^*$$

where

$$b_i^* = -2\sum_{j=1}^{n} \frac{\partial \tilde{a}_{ij}}{\partial x_j} - \tilde{b}_i \qquad (i=1,2,\dots,n),$$

$$c^* = \tilde{c} - \sum_{i=1}^{n} \frac{\partial \tilde{b}_i}{\partial x_i} - \sum_{i,j=1}^{n} \frac{\partial^2 \tilde{a}_{ij}}{\partial x_i \partial x_j}.$$

It is easy to verify that

$$\int_\Omega (\tilde{L}u)\, v\, dx = \int_\Omega u(\tilde{L}^* v)\, dx \tag{50}$$

for any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$, any $v \in H^{2,p'}(\Omega) \cap H_0^{1,p'}(\Omega)$ and $p' = p(p-1)^{-1}$. Since $1<p<2$ it follows $2<p'<+\infty$; applying lemma 1 to the operator $\tilde{L}^*$ we get the existence of a constant λ_0^* depending on ν, p, n and the coefficients of $\tilde{L}^*$ such that

$$\|v\|_{L_{p'}(\Omega)} \leqslant (\lambda - \lambda_0^*)^{-1} \|\tilde{L}^* v + \lambda v\|_{L_{p'}(\Omega)} \tag{51}$$

whenever $\lambda > \lambda_0^*$ and $v \in H^{2,p'}(\Omega) \cap H_0^{1,p'}(\Omega)$. From corollary 1 and theorem 1 the Dirichlet problem

$$\begin{cases} \tilde{L}^* v + \lambda v = g & \text{a.e. in } \Omega, \\ v \in H^{2,p'}(\Omega) \cap H_0^{1,p'}(\Omega) \end{cases}$$

has one and only one solution whenever $g \in L_{p'}(\Omega)$ and $\lambda \geq \hat{\lambda}^* = 2 + 3\lambda_0^*$. Now take any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$: since $|u|^{p-1} \operatorname{sign} u \in L_{p'}(\Omega)$, there exists one and only one solution w of the Dirichelet problem

$$(52) \qquad \begin{cases} \tilde{L}^* w + \lambda w = |u|^{p-1} \operatorname{sign} u & \text{a.e. in } \Omega, \\ w \in H^{2,p'}(\Omega) \cap H_0^{1,p'}(\Omega) \end{cases}$$

as soon as $\lambda \geq \hat{\lambda}^*$. It follows from (50), (51), (52) and Hölder's inequality:

$$(53) \qquad \int_\Omega |u|^p\, dx = \int_\Omega u(|u|^{p-1} \operatorname{sign} u)\, dx = \int_\Omega u(\tilde{L}^* w + \lambda w)\, dx =$$

$$= \int_\Omega (\tilde{L}u + \lambda u) w\, dx \leq \|\tilde{L}u + \lambda u\|_{L_p(\Omega)} \|w\|_{L_{p'}(\Omega)} \leq$$

$$\leq (\lambda - \lambda_0^*)^{-1} \|\tilde{L}u + \lambda u\|_{L_p(\Omega)} \|\tilde{L}^* w + \lambda w\|_{L_{p'}(\Omega)} = (\lambda - \lambda_0^*)^{-1} \|\tilde{L}u + \lambda u\|_{L_p(\Omega)} \|u\|^{p-1}_{L_p(\Omega)}.$$

From (53) it is easy to find

$$\|u\|_{L_p(\Omega)} \leq (\lambda - \lambda_0^*)^{-1} \|\tilde{L}u + \lambda u\|_{L_p(\Omega)}$$

for any $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$ and any $\lambda > \lambda_0^*$. ∎

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