[5]

Estratto da:
Rend. Ist. di Matem. Univ. di Trieste
Vol. 4, fasc. IV (1972).

THIRD BOUNDARY VALUE PROBLEM IN $H^{2,p}(\Omega)$ FOR A CLASS OF LINEAR SECOND ORDER ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS (*)

by MAURIZIO CHICCO (in Genova) (**)

SOMMARIO. - *Si studiano certi problemi al contorno di derivata obliqua per una classe di equazioni differenziali alle derivate parziali lineari ellittiche del secondo ordine, in cui i coefficienti delle derivate seconde sono (uniformemente) continui e gli altri appartengono ad opportune classi di sommabilità.*

SUMMARY. - *Some oblique derivative boundary value problems for a class of linear second order elliptic partial differential equations are studied. The coefficients of the second derivatives are supposed to be uniformly continuous and the other ones to belong to suitable L_p classes.*

1. Introduction.

The present work is, in a certain sense, the natural continuation of [3]. We consider the elliptic operator

$$(1) \qquad L = -\sum_{j,\, i=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c$$

where the coefficients a_{ij} are uniformly continuous in the open set Ω and the other ones belong to suitable $L_q(\Omega)$ classes. Given $f \in L_p(\Omega)$, with $1 < p < +\infty$, we look for sufficient conditions to

(*) Pervenuto in Redazione il 29 gennaio 1972.
Lavoro eseguito nell'ambito del « Centro di Matematica e Fisica Teorica del C. N. R. » presso l' Università di Genova.

(**) Indirizzo dell'Autore: Istituto di Matematica dell' Università — Via L. B. Alberti 4 — 16132 Genova.

solve the boundary value problem

$$(2) \qquad \begin{cases} Lu = f & \text{a. e. in } \Omega, \\ u \in H^{2,p}(\Omega) & \\ \sum_{i=1}^{n} \beta_i u_{x_i} + \gamma u = 0 & \text{on } \partial\Omega. \end{cases}$$

In the same way as in [3] for the Dirichlet problem, we prove that if $\text{ess}_\Omega \inf c > 0$ and $\gamma \geq 0$ (or also if $p > n$, $c \geq 0$, $\gamma > 0$) problem (2) has one and only one solution. The plan of the present work is very similar to that of [3]; I must nevertheless mention that here the important results of [1], [2], [6], ... are used, unlike in [3].

2. Notations and hypotheses.

In the following we shall always assume these hypotheses without mention. Let Ω an open bounded set in R^n, with $n \geq 3$ (for the case $n = 2$ see e. g. [9]). We suppose that the boundary of Ω (denoted by $\partial\Omega$) can be represented locally by a function with continuous second derivatives. Let p be a fixed real number greater than 1; we put, for shortness:

$$\| u_x \|_{L_p(\Omega)} = \sum_{i=1}^{n} \| u_{x_i} \|_{L_p(\Omega)} ; \quad \| u_{xx} \|_{L_p(\Omega)} = \sum_{i,j=1}^{n} \| u_{x_i x_j} \|_{L_p(\Omega)} .$$

We denote by $H^{1,p}(\Omega)$, $H^{2,p}(\Omega)$ the real Banach spaces obtained by completing $C^2(\bar{\Omega})$ according to the norms

$$(3) \qquad \| u \|_{H^{1,p}(\Omega)} = \| u \|_{L_p(\Omega)} + \| u_x \|_{L_p(\Omega)} ;$$

$$(4) \qquad \| u \|_{H^{2,p}(\Omega)} = \| u \|_{H^{1,p}(\Omega)} + \| u_{xx} \|_{L_p(\Omega)} .$$

Let $\nu \equiv (\nu_1, \nu_2, \ldots, \nu_n)$ denote the outward normal unit vector to $\partial\Omega$; let $\beta_i\ (i = 1, 2, \ldots, n)$, γ be functions such that $\beta_i, \gamma \in C^1(\partial\Omega)$ $(i = 1, 2, \ldots, n)$,

$$\sum_{i=1}^{n} \beta_i^2 \equiv 1 \quad \text{on} \quad \partial\Omega, \qquad \sum_{i=1}^{n} \nu_i \beta_i > 0 \quad \text{on} \quad \partial\Omega.$$

We denote by V the space

$$V = \text{completion in } H^{2,p}(\Omega) \text{ of}$$

$$\left\{ v : v \in C^2(\overline{\Omega}),\ \sum_{i=1}^{n} \beta_i v_{x_i} + \gamma v = 0 \ \text{ on } \partial\Omega \right\}.$$

Let a_{ij}, b_i, c be real valued functions defined in Ω such that:

$$a_{ij} \in C^0(\overline{\Omega}),\ a_{ij} = a_{ji}\,(i, j = 1, 2, \dots, n),\ \sum_{i,j=1}^{n} a_{ij} t_i t_j \geq m_0 \,|\, t \,|^2$$

in Ω, m_0 a positive constant, $b_i \in L_r(\Omega)$ $(i = 1, 2, \dots, n)$, $c \in L_s(\Omega)$ with $r = n$ if $1 < p < n$, $r > n$ if $p = n$, $r = p$ if $p > n$; $s = n/2$ if $1 < p < n/2$, $s > n/2$ if $p = n/2$, $s = p$ if $p > n/2$. Let L be the operator defined in (1). From the previous hypotheses and from known theorems on the spaces $H^{2,p}(\Omega)$ (see e. g. [5]) it follows that L is a bounded operator from $H^{2,p}(\Omega)$ in $L_p(\Omega)$ and in particular from V in $L_p(\Omega)$.

3. Preliminary lemmas.

Lemma 1. *For any $\varepsilon > 0$ there exists two positive constants K_1 and λ_0 (K_1 depending on Ω, n, p, β_i, γ, $a_{ij}\,(i, j = 1, 2, \dots, n)$ and λ_0 also on ε, b_i, c $(i = 1, 2, \dots, n)$) such that for any $u \in V$ and uniformly for any $\lambda \geq \lambda_0$ it turns out*

$$\| u \|_{L_p(\Omega)} \leq K_1 \lambda^{-1} \| Lu + \lambda u \|_{L_p(\Omega)} + \varepsilon \lambda^{-1} \| u \|_{H^{2,p}(\Omega)} . \tag{5}$$

Proof. From the hypotheses it follows that for example theorem 4.1 of [6] or theorem 2.1 of [1] can be applied, obtaining the existence of two constants K_2 and m_1, depending on Ω, n, p, β_i, γ, a_{ij} $(i, j = 1, 2, \dots, n)$, such that

$$\| u \|_{L_p(\Omega)} \leq K_2 \lambda^{-1} \left\| \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \lambda u \right\|_{L_p(\Omega)} \tag{6}$$

for any $u \in V$ and uniformly for any $\lambda \geq m_1$. From known properties of the space $H^{2,p}(\Omega)$ (see e. g. [3] lemma 3) for any $\eta > 0$ there exists a constant K_3 depending on η, b_i, c, p, n, Ω such that

$$\left\| \sum_{i=1}^{n} b_i u_{x_i} + cu \right\|_{L_p(\Omega)} \leq \eta \| u \|_{H^{2,p}(\Omega)} + K_3 \| u \|_{L_p(\Omega)} \tag{7}$$

for any $u \in H^{2,p}(\Omega)$. Let us fix an arbitrary $\varepsilon > 0$ and choose $\eta = \varepsilon(2K_2)^{-1}$ in (7). So from (6), (7) it follows

$$(8) \quad \| u \|_{L_p(\Omega)} \leq \frac{K_2}{\lambda} \| Lu + \lambda u \|_{L_p(\Omega)} + \frac{\varepsilon}{2\lambda} \| u \|_{H^{2,p}(\Omega)} + \frac{K_2 K_3}{\lambda} \| u \|_{L_p(\Omega)}$$

valid for any $u \in V$ and any $\lambda \geq m_1$. If now in (8) we choose $\lambda \geq \lambda_0 = \max(m_1, 2K_2K_3)$, we get easily

$$(9) \quad \| u \|_{L_p(\Omega)} \leq \frac{2K_2}{\lambda} \| Lu + \lambda u \|_{L_p(\Omega)} + \frac{\varepsilon}{\lambda} \| u \|_{H^{2,p}(\Omega)}$$

for any $u \in V$ and any $\lambda \geq \lambda_0$. ∎

LEMMA 2. *There exists a positive constant K_4 depending on Ω, n, p and the coefficients of L such that*

$$(10) \quad \| u \|_{H^{2,p}(\Omega)} \leq K_4 \{ \| Lu \|_{L_p(\Omega)} + \| u \|_{L_p(\Omega)} \}$$

for any $u \in V$.

PROOF. From the fundamental results of [2] it turns out

$$(11) \quad \| u \|_{H^{2,p}(\Omega)} \leq K_5 \left\{ \left\| \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} \right\|_{L_p(\Omega)} + \| u \|_{L_p(\Omega)} \right\}$$

for any $u \in V$, where K_5 depends on p, n, Ω, β_i, γ, a_{ij} $(i, j = 1, 2, \ldots, n)$. From (7), (11) we deduce

$$(12) \quad \| u \|_{H^{2,p}(\Omega)} \leq K_5 \{ \| Lu \|_{L_p(\Omega)} + \eta \| u \|_{H^{2,p}(\Omega)} + (K_3 + 1) \| u \|_{L_p(\Omega)} \}$$

for any $u \in V$ and any $\eta > 0$. It is sufficient to choose in (12) $\eta = (2K_5)^{-1}$ and (10) is reached with $K_4 = 2K_5(K_3 + 1)$. ∎

4. Main results.

THEOREM 1. *There exists two positive constants $\hat{\lambda}$, K_6 depending on n, p, β_i, γ, Ω and the coefficients of L such that*

$$(13) \quad \| u \|_{H^{2,p}(\Omega)} \leq K_6 \| Lu + \lambda u \|_{L_p(\Omega)}$$

for any $u \in V$ and uniformly for any $\lambda \geq \hat{\lambda}$.

PROOF. From (10) it follows

$$\| u \|_{H^{2,p}(\Omega)} \leq K_4 \{ \| Lu + \lambda u \|_{L_p(\Omega)} + (\lambda + 1) \| u \|_{L_p(\Omega)} \} \tag{14}$$

valid for any $u \in V$ and any $\lambda \geq 0$.

From (14) and lemma 1 we get

$$\| u \|_{H^{2,p}(\Omega)} \leq \tag{15}$$

$$\leq K_4 \left\{ \left(1 + \frac{\lambda + 1}{\lambda} K_1 \right) \| Lu + \lambda u \|_{L_p(\Omega)} + \frac{\lambda + 1}{\lambda} \varepsilon \| u \|_{H^{2,p}(\Omega)} \right\}$$

for any $u \in V$ and uniformly for any $\lambda \geq \lambda_0$.

By choosing in (15) $\varepsilon = (4K_4)^{-1}$ it turns out

$$\| u \|_{H^{2,p}(\Omega)} \leq 2K_4 (1 + 2K_1) \| Lu + \lambda u \|_{L_p(\Omega)} \tag{16}$$

valid for any $u \in V$ and any $\lambda \geq \max(\lambda_0, 1)$. ▮

COROLLARY 1. *Let f be given in $L_p(\Omega)$, let $\lambda \geq \widehat{\lambda}$ where $\widehat{\lambda}$ is the constant introduced in the preceding theorem. Then the boundary value problem*

$$\begin{cases} Lu + \lambda u = f & \text{a. e. in } \Omega, \\ u \in V \end{cases} \tag{17}$$

has one and only one solution.

PROOF. Let $L^{(m)}$ $(m = 1, 2, \ldots)$ be the operators defined by

$$L^{(m)} = - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} b_i^{(m)} \frac{\partial}{\partial x_i} + c^{(m)} \tag{18}$$

where $b_i^{(m)}, c^{(m)} \in C^\infty(\overline{\Omega})$ $(i = 1, 2, \ldots, n)$ and

$$\lim_{m \to +\infty} \left\{ \sum_{i=1}^{n} \| b_i^{(m)} - b_i \|_{L_r(\Omega)} + \| c^{(m)} - c \|_{L_s(\Omega)} \right\} = 0. \tag{19}$$

It is easy to check that (13) is verified for the operators $L^{(m)}$ also, with K_6 and $\widehat{\lambda}$ independent on m. Besides from the results of [1] (theorem 2.1) and [6] (theorem 4.1) the operators $L^{(m)}$ have non emptʎ resolvent set, therefore countable spectrum. From these facts and from (13) (applied to the operators $L^{(m)}$) it follows that the boundary value problems

$$(20)\qquad \begin{cases} L^{(m)} u^{(m)} + \lambda u^{(m)} = f & \text{a. e. in } \Omega, \\ u^{(m)} \in V \end{cases} \qquad (m = 1, 2, \ldots)$$

have one and only one solution $u^{(m)}$ as soon as $\lambda \geq \hat{\lambda}$ and $f \in L_p(\Omega)$. Moreover it turns out

$$(21)\qquad \| u^{(m)} \|_{H^{2,p}(\Omega)} \leq K_6 \| f \|_{L_p(\Omega)} \qquad (m = 1, 2, \ldots)$$

with $\hat{\lambda}$ and K_6 independent on m.

From (21) there exists a sequence extracted from $\}u^{(m)}\{_{m \in N}$ which converges weakly in $H^{2,p}(\Omega)$ to a function $u \in V$. It is easy to verify through (20) that u is a solution of problem (17); its uniqueness is immediate from (13). ▮

The next theorem is the main result of the work; we suppose temporarily $p > n$.

THEOREM 2. *Suppose* $p > n$, $\operatorname{ess\,inf}_{\Omega} c \geq 0$, $\min_{\partial\Omega} \gamma \geq 0$, $\operatorname{ess\,inf}_{\Omega} c + \min_{\partial\Omega} \gamma > 0$, $f \in L_p(\Omega)$. *Then the boundary value problem*

$$(22)\qquad \begin{cases} Lu = f & \text{a. e. in } \Omega, \\ u \in V \end{cases}$$

has one and only one solution. Moreover if $\operatorname{ess\,inf}_{\Omega} f < 0$, $\operatorname{ess\,sup}_{\Omega} f \leq 0$ *it turns out* $\max_{\bar{\Omega}} u < 0$.

PROOF. We begin to show that if $f \leq 0$ a. e. in Ω it follows $u \leq 0$ a. e. in Ω, whence the uniqueness of the solution u. From corollary 1 the operator L has non-empty resolvent set and therefore its spectrum is discrete and countable. So from the Riesz-Fredholm theory the uniqueness of the solution u will imply its existence.

Since $u \in H^{2,p}(\Omega)$ and $p > n$, for known properties of the space $H^{2,p}(\Omega)$ (see e. g. [5]) the first derivatives of u exist in every point of Ω (and are Hölder continuous in $\bar{\Omega}$). This is sufficient to apply the maximum principle: I give a sketch of proof for the sake of completeness.

Let us put $M = \max_{\bar{\Omega}} u$ and suppose $M \geq 0$ in order to find a contradiction. First of all, if $u \equiv M$ in Ω it follows $M = 0$ and the result is proven. If u is not identically equal to M but $u(x) = M$ for some $x \in \Omega$, then it is possible to find an (open) ball S contained in $\}x : x \in \Omega,\ u(x) < M\{$ such that there exists $x_0 \in \Omega \cap \partial S$ with $u(x_0) = M$. This is a contradiction because obviously $u_{x_i}(x_0) = 0$ $(i = 1, 2, \dots, n)$ while from known results (see e. g. [7] page 67) it turns out $\left(\frac{\partial u}{\partial l}\right)(x_0) > 0$ where l is the outward normal direction to ∂S in x_0.

Therefore the maximum M is attained only in points of $\partial \Omega$. If $\bar{x} \in \partial \Omega$ is such that $u(\bar{x}) = M$, since $u \in V$ we have

$$\sum_{i=1}^{n} \beta_i(\bar{x})\, u_{x_i} + \gamma(\bar{x})\, M = 0. \tag{23}$$

As $M \geq 0$, it follows $\sum_{i=1}^{n} \beta_i(\bar{x})\, u_{x_i}(\bar{x}) \leq 0$, a contradiction for the already mentioned results ([8], [7] page 67): in fact any outward derivative of u in $\bar{x}$ must be strictly positive. So we get $M < 0$.

In conclusion from the hypothesis $f \leq 0$ a. e. in Ω we have deduced that one of the following alternatives is satisfied:

(i) $\quad u \equiv 0 \quad$ in Ω;

(ii) $\quad \max_{\bar{\Omega}} u < 0.$

Since $u = 0$ a. e. in Ω implies $f = 0$ a. e. in Ω, the last assertion of the theorem easily follows. ▒

In the following corollary the assumption $p > n$ is dropped.

COROLLARY 2. *Suppose* $\operatorname{ess\,inf}_{\Omega} c > 0$, $\min_{\partial\Omega} \gamma \geq 0$, $f \in L_p(\Omega)$ $(1 < p < +\infty)$. *Then there exists one and only one solution* u *of the boundary value problem* (22). *Besides if* $f \leq 0$ *a e. in* Ω *it follows* $u \leq 0$ *a. e. in* Ω.

PROOF. Let $\}L^{(m)}\{_{m \in N}$ be the sequence of operators defined in (18), (19). It is easy to verify (see e. g. lemma 2 of [3]) that

$$\lim_{m \to +\infty} \max \} \| Lw - L^{(m)} w \|_{L_p(\Omega)} : \| w \|_{H^{2,p}(\Omega)} \leq 1 \{ = 0. \tag{24}$$

From (24) one deduces that the eigenvalues of $\{L^{(m)}\}_{m \in N}$ converge, when $m \to +\infty$, to the respective eigenvalues of L (see e. g. [4] page 1091). Let us prove now that the real numbers less than $\operatorname{ess\,inf}_{\Omega} c$ cannot be eigenvalues of any of the operators $L^{(m)}$ $(m = 1, 2, \ldots)$.

It will follow that the real eigenvalues of L also are not less than $\operatorname{ess\,inf}_{\Omega} c$, therefore 0 is not an eigenvalue of L.

So let us suppose that $\lambda < \operatorname{ess\,inf}_{\Omega} c$. Obviously the coefficients $\{c^{(m)}\}_{m \in N}$ of the operators $\{L^{(m)}\}_{m \in N}$ can be chosen in such a way that $c^{(m)} \geq \operatorname{ess\,inf}_{\Omega} c$ in $\bar{\Omega}$. Let $w \in V$ and $L^{(m)} w = \lambda w$ a. e. in Ω: in order to prove that 0 is not an eigenvalue of $L^{(m)}$ it is sufficient to show that $w = 0$ a. e. in Ω.

Since

$$(25) \qquad - \sum_{i, j=1}^{n} a_{ij}\, w_{x_i} + \sum_{i=1}^{n} b_i^{(m)}\, w_{x_i} + [c^{(m)} - \lambda]\, w = 0 \quad \text{a. e. in } \Omega$$

and $c^{(m)} - \lambda > 0$, theorem 2 can be applied: in fact $b_i^{(m)}, c^{(m)} \in C^{\infty}(\bar{\Omega})$ $(i = 1, 2, \ldots, n)$ and therefore $w \in H^{2,q}(\Omega)$ for all $q > 1$.

From theorem 2 we get $w = 0$ a. e. in Ω, so λ is not an eigenvalue of the operators $L^{(m)}$ $(m = 1, 2, \ldots)$. This means that the real eigenvalues of L are not less than $\operatorname{ess\,inf}_{\Omega} c$, whence 0 is not an eigenvalue of L and problem (22) has one and only one solution.

We have to prove now that if $f \leq 0$ a. e. in Ω it turns out $u \leq 0$ a. e. in Ω. Let $\{f_m\}_{m \in N}$ be a sequence of functions such that

$$(26) \qquad \begin{cases} f_m \in C^{\infty}(\bar{\Omega}), \quad f_m \leq 0 \text{ in } \bar{\Omega}, & (m = 1, 2, \ldots), \\ \lim\limits_{m \to +\infty} \| f - f_m \|_{L_p(\Omega)} = 0. \end{cases}$$

Denote by $\{u_m\}_{m \in N}$ the sequence of the solutions of the boundary value problems

$$(27) \qquad \begin{cases} L^{(m)} u_m = f_m \quad \text{in } \Omega, \\ u_m \in V \end{cases} \qquad (m = 1, 2, \ldots)$$

It is easy to check through (24), (26), (27) that

$$(28) \qquad \lim_{m \to +\infty} \| u - u_m \|_{L_p(\Omega)} = 0.$$

REFERENCES

[1] AGMON, S.: *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems.* Comm. Pure Appl. Math. 15 (1962), 119-147.

[2] AGMON, S.; DOUGLIS, A.; NIRENBERG, L.: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions,* I. Comm. Pure Appl. Math. 12 (1959), 623-727.

[3] CHICCO, M.: *Solvability of the Dirichlet problem in $H^{2,p}(\Omega)$ for a class of linear second order elliptic partial differential equations.* Boll. Un. Mat. Ital. (4) 4 (1971), 374-387.

[4] DUNFORD, N.; SCHWARTZ, J. T.: *Linear operators.* Interscience Publ., New York.

[5] GAGLIARDO, E.: *Proprietà di alcune classi di funzioni in più variabili.* Ricerche Mat. 7 (1958), 102-137.

[6] GEYMONAT, G.; GRISVARD, P.: *Alcuni risultati di teoria spettrale per i problemi ai limiti ellittici.* Rend. Sem. Mat. Univ. Padova 38 (1967), 121-173.

[7] PROTTER, M. H.; WEINBERGER, M. H.: *Maximum principles in differential equations.* Prentice Hall Publ., Englewood Cliffs (1967).

[8] PUCCI, C.: *Proprietà di massimo e minimo delle soluzioni di equazioni a derivate parziali del secondo ordine di tipo ellittico e parabolico.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) 23 (1957), 370-375 and (8) 24 (1958), 3-6.

[9] TALENTI, G.: *Problemi di derivata obliqua per equazioni ellittiche in due variabili.* Boll. Un. Mat. Ital. (3) 22 (1967), 505-526.