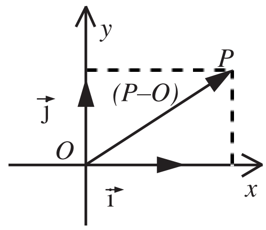
Analytic geometry

Cartesian coordinates

We recall the definition of cartesian coordinate system.
In the space (in the plane) fix an orthonormal basis $\vec{\imath}, \vec{\jmath}, \vec{k}$ ($\vec{\imath}, \vec{\jmath}$) of vectors of V_3 (of V_2) and a point O. This choice defines a cartesian monometric orthogonal coordinate system.
More, if the basis $\vec{\imath}, \vec{\jmath}, \vec{k}$ is right-handed, the system too is called right-handed.

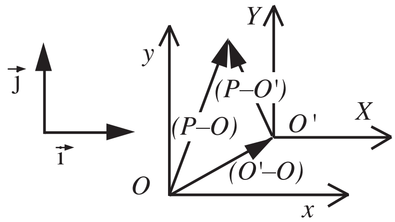


If P is a point in the space (in the plane), the cartesian coordinates of P are the coordinates of the vector $(P - O)$ with respect to the choosen o.n. basis.
To be clear: if $(P{-}O) = x\vec{\imath}{+}y\vec{\jmath}{+}z\vec{k}$, then the cartesian coordinates of P are (x, y, z). This way of defining cartesian coordinates is very useful in changes of coordinates.

Changes of coordinates

If we have two coordinate systems it is important to see the relation between the coordinates of one point in the two systems and how representations of objects vary.
There are three cases:

Translation: If we have two sistems $Oxyz$ and $O'XYZ$ which have different origin and the same o.n. basis, this is a translation. Let $\{x = x_0\ ;\ y = y_0\ ;\ z = z_0\}$ be the coordinates of O' with respect to the system $Oxyz$.



Since $(P - O') = (P - O) - (O' - O)$, then the relation between the coordinates (x, y, z) of a point with respect to Oxy e coordinates (X, Y, Z) is:

$$\begin{cases} X &= x - x_0 \\ Y &= y - y_0 \\ Z &= z - z_0 \end{cases}$$

Rotation: If we have two sistems $Oxyz$ and $OXYZ$ which have the same origin, but different bases $\vec{\imath}, \vec{\jmath}, \vec{k}$ and $\vec{\imath}', \vec{\jmath}', \vec{k}'$, this is a rotation.
Let (p_{11}, p_{21}, p_{31}) be the vector coordinates of $\vec{\imath}'$ with respect to $\vec{\imath}, \vec{\jmath}, \vec{k}$, that is $\vec{\imath}' = p_{11}\vec{\imath} + p_{21}\vec{\jmath} + p_{31}\vec{k}$.
Similarly, let (p_{12}, p_{22}, p_{32}) and (p_{13}, p_{23}, p_{33}) be respectively the vector coordinates of $\vec{\jmath}'$ and $\vec{k}'$ with respect to $\vec{\imath}, \vec{\jmath}, \vec{k}$, then the the relation between the coordinates (x, y, z) of a point with respect to Oxy and the coordinates (X, Y, Z) is: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

Note that the columns of the matrix $P = (p_{ij})$ are the vector coordinates of $\vec{\imath}', \vec{\jmath}'$ and $\vec{k}'$ with respect to $\vec{\imath}, \vec{\jmath}, \vec{k}$.
The matrix P is orthogonal and has determinant 1, that is, it is a rotation matrix.

The inverse change is $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} p_{11} & p_{21} & p_{31} \\ p_{12} & p_{22} & p_{32} \\ p_{13} & p_{23} & p_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
and the rotation matrix is $P^T = P^{-1}$.

Rototranslation: If both origin and basis change, the we have a rototranslation which can be considered as a composition of a translation and a rotation (in some cases a composition of a rotation and a translation).
The relations are easily written if we know the vector coordinates of $\vec{\imath}', \vec{\jmath}'$ and $\vec{k}'$ with respect to $\vec{\imath}, \vec{\jmath}, \vec{k}$ and the coordinates of O' with respect to the system $Oxyz$.

$$\begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \qquad \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} p_{11} & p_{21} & p_{31} \\ p_{12} & p_{22} & p_{32} \\ p_{13} & p_{23} & p_{33} \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix}$$

Curves in the plane

It's not easy to give a good definition of curve. There is plenty of definitions and one can use the one which is more suitable for his needs.
We can only say that, if $f(x, y)$ is a "good" function, that is continuous (or almost so), differentiable (or almost so), defined somewhere in the plane, then the set of (x, y) such that

$$f(x, y) = 0$$

in most cases is a set which corresponds to the naive idea of (cartesian) curve.

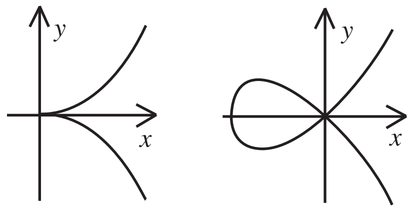
Furthermore, if $x(t), y(t)$ are "good" functions, and I is some "good" subset of $\mathbb{R}$, usually an interval, then the set of (x, y) obtained this way:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad t \in I$$

in most cases is a set which corresponds to the naive idea of (parametric) curve.
Depending on the current problem, sometimes it is more convenient to deal with a parametric curve, sometimes with a cartesian curve. So we have this problem.
Is it possible to transform a parametric curve into a cartesian one and viceversa? The answer is not easy, but as a general matter of facts we can say:
It is often possible to transform a parametric representation into a cartesian one, but the obtained curve is sometimes bigger and computations are often very hard or require non-elementary functions.
The viceversa is in general even harder. Often non-elementary functions are needed and the resulting curve may be smaller. In most cases there is not a practical way to carry out the computation.

Example 1: $y - x^2 = 0$ is a simple parabola. In this case $\{x = t \ ; \ y = t^2\}$ $t \in \mathbb{R}$ is a parametric reprentation which is exactly the same curve.

Example 2: $x^2 + y^2 - 1 = 0$ is a circle. In this case there are many ways to write a parametric representation.
The simplest is $\{x = \cos(t) \ ; \ y = \sin(t)\}$ $t \in [0, 2\pi)$, but there is a lack of continuity.
Or you can write $\{x = \cos(t) \ ; \ y = \sin(t)\}$ $t \in \mathbb{R}$, but every point is obtained infinite times.
Another parametric representation of the circle is $\left\{ x = \dfrac{1 - t^2}{1 + t^2} \ ; \ y = \dfrac{2t}{1 + t^2} \right\}$ $t \in \mathbb{R}$
In this case the point $(-1, 0)$ is missing and representation is not accurate in the neighbourhood of this point.



Example 3 $\{x = t^2 \ ; \ y = t^3\}$ $t \in \mathbb{R}$. It is cubic curve with a cusp.
A simple trick allows us to get a cartesian equation:

$$\{x^3 = t^6 \ ; \ y^2 = t^6\} \qquad \rightarrow \qquad x^3 = y^2$$

Example 4 $\{x = t^2 - 1 \ ; \ y = t(t^2 - 1)\}$ $t \in \mathbb{R}$. It is a cubic curve with a knot. Note that the knot point $(0, 0)$ is obtained twice $(t = \pm 1)$. As above, there is a trick to get a cartesian equation:

$$\{x = t^2 - 1 \ ; \ y = t\,x\} \ \rightarrow \ y^2 = t^2 x^2 \ \rightarrow \ y^2 = (x + 1)x^2$$

Curves and surfaces in the space
As for the curves, there is not a universal definition of surface.
We can only say that, if $f(x, y, z)$ is a "good" function like above, then the set of (x, y, z) such that

$$f(x, y, z) = 0$$

in most cases is a set which corresponds to the naive idea of (cartesian) surface.
Furthermore, if $x(u, v), y(u, v), z(u, v)$ are "good" functions, and D is some "good" domain in $\mathbb{R}^2$, then the set of (x, y, z) obtained this way:

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases} \qquad (u, v) \in D$$

in most cases is a set which corresponds to the naive idea of (parametric) surface.

Passing from parametric representation to cartesian one and viceversa involves the same kind of problems as for curves in the plane.
Often the intersection of two surfaces is a curve in the space, and so the set of (x, y, z) such that

$$\begin{cases} f(x, y, z) = 0 \\ g(x, y, z) = 0 \end{cases}$$

in most cases is a (cartesian) curve.
On the other hand

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \qquad t \in I$$

in most cases is a (parametric) curve.

Conics

If $f(x, y)$ is a second degree polynomial, then the the set of (x, y) such that $f(x, y) = 0$ is call conic (or conic section). In most cases a conic is a curve, although there are some exceptions.
The general equation is

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

The two symmetric matrices $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$ $B = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$ give some information about the conic. Namely:

If $\det(B) > 0$ the conic is of elliptic kind
If $\det(B) = 0$ the conic is of parabolic kind
If $\det(B) < 0$ the conic is of hyperbolic kind

$\det(A) = 0$ if and only if the conic is degenerated

There are 9 kinds of conics We give a brief account of classification of conics in the following table.

Canonical eq.	Description	Kind	Center	degenerated
$ax^2 = 0$	two coincident lines	parabolic	No	Yes
$a^2x^2 = 1$	two parallel lines	parabolic	No	Yes
$a^2x^2 = -1$	two parallel lines without real points	parabolic	No	Yes
$y = ax^2$	parabola	parabolic	No	No
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$	two intersecting lines	hyperbolic	Yes	Yes
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	hyperbola	hyperbolic	Yes	No
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$	only one real point	elliptic	Yes	Yes
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$	ellipse without real points	elliptic	Yes	No
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	ellipse	elliptic	Yes	No

A brief account of some tools useful in studying conics:

• The center of the conic (when it exists) is the solution of the linear system of partial derivatives of the equation with respect to x and y. If we divide the derivatives by 2 the system is:

$$\begin{cases} a_{11}x + a_{12}y + a_{13} = 0 \\ a_{12}x + a_{22}y + a_{23} = 0 \end{cases}$$

• To decide whether an ellipse has real points, one can intersect the conic with a line poassing through its center (by instance the line $a_{11}x + a_{12}y + a_{13} = 0$).

• Let γ be a conic having center C, then it has two symmetry axes. Let λ_1 e λ_2 be the two eigenvalues of the matrix B ($\lambda_1 \neq \lambda_2$, if γ is not a circle) and let $\vec{v}_1, \vec{v}_2$ be two respective eigenvectors (note that $\vec{v}_1 \cdot \vec{v}_2 = 0$ since B is symmetric).
Then the lines passing through C and having the directions of $\vec{v}_1$ and $\vec{v}_2$ are the axes.

• If the conic is of hyperbolic kind, the quadratic form of its equation can be decomposed into a product of two linear forms:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = (px + qy)(p'x + q'y)$$

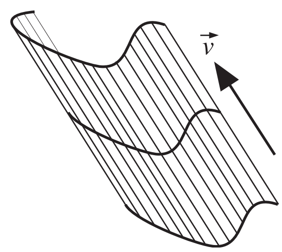
Then the asymptotes of the conic are *parallel* to the lines $px+qy=0$; $p'x+q'y=0$ and pass through C.

Ruled surfaces

A surface which is a locus of lines is called a ruled surface (it is also called a scroll).
This means that a surface $\mathcal{S}$ is ruled if through every point of $\mathcal{S}$ there is a straight line that lies on S.

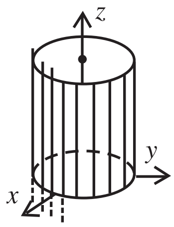
Cylinders

A surface which is a locus of lines which are all mutually parallel is called a (generalised) cylinder. The lines are called generatrices. A curve meeting all the generatrices is called a directrix.
This means that, if $\mathcal{S}$ is a cylinder, then there is a non-null vector $\vec{v}$ such that, given any point $P \in \mathcal{S}$, then the line passing through P and having the same direction as $\vec{v}$ is entirely contained in $\mathcal{S}$.

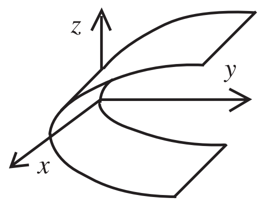


The simplest cylinders are those whose direction vector is $\vec{\imath}$ or $\vec{\jmath}$ or $\vec{k}$.

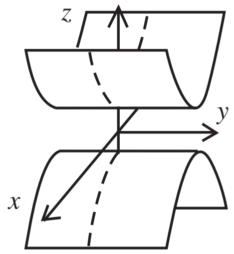
In fact the set $\mathcal{S}$ of points **in space** described by the equation $f(x,y)=0$ (not depending on z) is a cylinder, since if $P(x_0,y_0,z_0)$ is a point of $\mathcal{S}$, then all the points $P(x_0,y_0,z)$ (that is the points of the line through P and parallel to $\vec{k}$) lie on $\mathcal{S}$.
The same arguments applies to surfaces $f(y,z)=0$ (cylinder parallel to $\vec{\imath}$) and $f(x,z)=0$ (cylinder parallel to $\vec{\jmath}$). Some simple examples:



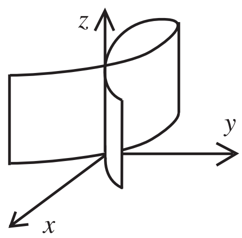
$x^2+y^2=1$



$y=z^2$



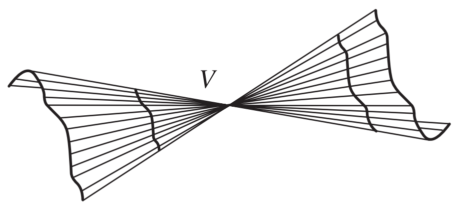
$z^2-x^2=1$



$y^2=x^2(x+1)$

Cones

A surface which is a locus of lines which are all passing through the same point V (called vertex) is called a (generalised) cone. The lines are called generatrices. A curve meeting all the generatrices is called a directrix.
This means that, if $\mathcal{S}$ is a cone, then there is a point V such that, given any point $P \in \mathcal{S}$, then the line passing through P and V is entirely contained in $\mathcal{S}$.

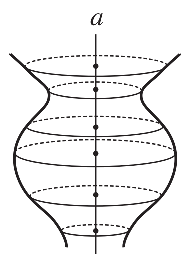


The simplest cones are those whose vertex is the point O.

We recall that a function $f(x_1,...,x_n)$ is called homogeneous of degree α if $f(tx_1,...,tx_n)=t^\alpha f(x_1,...,x_n)$ for any $t \in \mathbb{R}$. By example a polynomial $P(x_1,...,x_n)$ is homogeneous if and only if its monomials have the same degree.
The set $\mathcal{S}$ of points described by the equation $f(x,y,z)=0$, where $f(x,y,z)$ is homogeneous, is a cone whose vertex is $O(0,0,0)$.
In fact, if (x_0,y_0,z_0) is a point of $\mathcal{S}$, then the line through P and O is formed by the points (tx_0,ty_0,tz_0) ($t \in \mathbb{R}$) and, since f is homogeneous, then $f(tx_0,ty_0,tz_0)=t^\alpha f(x_0,y_0,z_0)=0$, that is they all lie on $\mathcal{S}$.

Surfaces of revolution

A surface of revolution $\mathcal{S}$ is a surface which is a locus of circles all having the same axis.
This means that there is a line a (called axis). Given any point $P \in \mathcal{S}$, consider the circle which lies in a plane orthogonal to the axis, has center on the axis and passes through P. Then this circle is entirely contained in $\mathcal{S}$.



Construction of cones, cylinders and surfaces of revolution

Let $\mathcal{L}:\{x=x(t)\ ;\ y=y(t)\ ;\ z=z(t)\}$ be a curve.
If $\vec{v}\,(l,m,n)$ is a non-null vector, the the cylinder having $\mathcal{L}$ as directrix and with generatices parallel to

$\vec{v}$ has this parametric representation

$$\begin{cases} x = x(t) + l\,u \\ y = y(t) + m\,u \\ z = z(t) + n\,u \end{cases}$$

If $V(x_0, y_0, z_0)$ is a point, the the cone having $\mathcal{L}$ as directrix and V as vertex has this parametric representation

$$\begin{cases} x = x_0 + x(t) \cdot u \\ y = y_0 + y(t) \cdot u \\ z = z_0 + z(t) \cdot u \end{cases}$$

If a is a line, the surface of revolution containing $\mathcal{L}$ and having a as axis can be constructed by considering for each point P of $\mathcal{L}$ the circle passing through P and having a as axis.
When a is the z axis, the surface has this simple parametric representation

$$\begin{cases} x = x(t) \cdot \cos(u) \\ y = y(t) \cdot \sin(u) \\ z = z(t) \end{cases}$$

Quadrics

If $f(x, y, z)$ is a second degree polynomial, then the the set of (x, y, z) such that $f(x, y, z) = 0$ is call quadric. In most cases a quadric is a surface, although there are some exceptions, as we shall see. The general equation is

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0$$

The two symmetric matrices $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}$ $\quad B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

give some information about the quadric, but things are more complicated than for conics.

There are 17 kinds of quadrics. It follows a brief description of all of them.

Quadric cylinders

1, 2, 3, ..., 9. The first 9 quadrics are cylinders of the respective conics.

Examples:
$y = x^2$ is a parabolic cylinder.

$x^2 - y^2 = 0$ is the cylinder of the degenerated hyperbola formed by two intersecting lines and so the quadric is formed by two intersecting planes.

$x^2 + y^2 = 0$ is only a line.

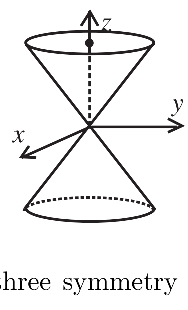
Then there are other 8 kinds of quadrics . We describe them in their canonical forms.

If we intersect a cylinder with a plane which is not parallel to generatrices we get always the same kind of conic. For instance, by intersecting an elliptic cylinder we get ellipses (and sometimes circles).

Quadric cones

10. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 0$. Though the equation is homogeneous, this is not a real cone, since the only real point is the vertex O. It is called degenerated cone or cone without real sheet.

11. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$. This is a true cone with the vertex in O.



The cone is a ruled surface. The lines can be written by using an algebraic trick:

$$\left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = -\left(\frac{y}{b}\right)\left(\frac{y}{b}\right) \quad \longrightarrow \quad \left(\frac{x}{a} - \frac{z}{c}\right) \Big/ \left(\frac{y}{b}\right) = -\left(\frac{y}{b}\right) \Big/ \left(\frac{x}{a} + \frac{z}{c}\right) = k$$

So the lines are $\begin{cases} \left(\dfrac{x}{a} - \dfrac{z}{c}\right) = k\left(\dfrac{y}{b}\right) \\ -\left(\dfrac{y}{b}\right) = k\left(\dfrac{x}{a} + \dfrac{z}{c}\right) \end{cases} \quad (k \in \mathbb{R})$ But one of the lines is missing: $\{y/b = 0\ ;\ x/a + z/c = 0\}$

In general the cone has one symmetry center (the vertex), three symmetry axes and three symmetry planes.

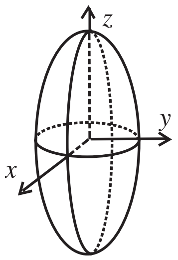
If $a = b$ the cone is a revolution surface (around z axis). In this case all the lines through O and orthogonal to z axis are symmetry axes and all the planes through z axis are symmetry planes.

Almost all the kinds of conics can be found by intersecting a cone with a plane. Parabolas are the intersections between the cone and planes wich are parallel to tangent planes.

Note: The first 11 quadrics are all degenerated and $\det(A) = 0$ (A the 4×4 matrix).

Non degenerated quadrics

12. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. This is an ellipsoid.

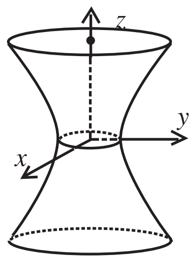


In general this quadric has one symmetry center, three symmetry axes and three symmetry planes.

If $a = b$ the ellipsoid is a revolution surface (around z axis). In this case all the lines through O and orthogonal to z axis are symmetry axes and all the planes through z axis are symmetry planes.
If $a = b = c$ the ellipsoid is a sphere and any plane and any line through O is of symmetry.

13. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = -1$. This is an ellipsoid without real points.
There is not much to say about this quadric.

14. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$. This is an hyperboloid of one sheet.



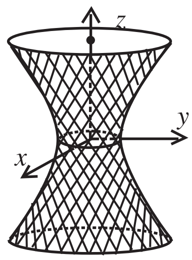
This quadric is doubly ruled. This means that through every point of $\mathcal{S}$ there are exactly two lines that lies on S. The lines are divided into two families. Each line of the first family does not intersect any line of the same family, but it does intersect any line of the other family and viceversa.

The two family of lines can be written by using an algebraic trick:

$$\left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right)\left(1 + \frac{y}{b}\right) \quad \longrightarrow \quad \left(\frac{x}{a} - \frac{z}{c}\right) \Big/ \left(1 - \frac{y}{b}\right) = \left(1 + \frac{y}{b}\right) \Big/ \left(\frac{x}{a} + \frac{z}{c}\right) = k$$

You can exchange the way of forming the fractions and then the families of lines are:

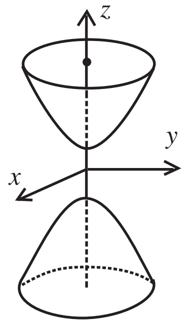
$$\begin{cases} \left(\frac{x}{a} - \frac{z}{c}\right) = k\left(1 - \frac{y}{b}\right) \\ \left(1 + \frac{y}{b}\right) = k\left(\frac{x}{a} + \frac{z}{c}\right) \end{cases} \quad \begin{cases} \left(\frac{x}{a} - \frac{z}{c}\right) = k\left(1 + \frac{y}{b}\right) \\ \left(1 - \frac{y}{b}\right) = k\left(\frac{x}{a} + \frac{z}{c}\right) \end{cases} \quad (k \in \mathbb{R})$$



Note that in these representations, like for the cone, two lines are missing.
In general this quadric has one symmetry center, three symmetry axes and three symmetry planes.

If $a = b$ the hyperboloid is a revolution surface (around z axis). In this case all the lines through O and orthogonal to z axis are symmetry axes and all the planes through z axis are symmetry planes.
Almost all the kinds of conics can be found by intersecting a hyperboloid of two sheets with a plane. Parabolas are the intersections between the quadric and planes which are parallel to planes which are tangent to the asymptotic cone $x^2/a^2 + y^2/b^c - z^2/c^2 = 0$.

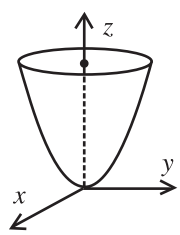
15. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$. This is an hyperboloid of two sheets.



This is not a ruled surface.
In general this quadric has one symmetry center, three symmetry axes and three symmetry planes.

If $a = b$ the hyperboloid is a revolution surface (around z axis). In this case all the lines through O and orthogonal to z axis are symmetry axes and all the planes through z axis are symmetry planes.

16. $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$. This is an elliptic paraboloid.
This is not a ruled surface.

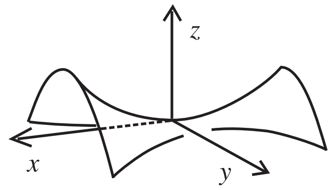


This quadric has no symmetry center, one symmetry axis and two symmetry planes.
If $a = b$ the paraboloid is a revolution surface (around z axis). In this case all the planes through z axis are symmetry planes.

The elliptic paraboloid does not contain hyperbolas.

17. $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$. This is an elliptic paraboloid.

This quadric is doubly ruled. This means that through every point of $\mathcal{S}$ there are exactly two lines that lies on S. The lines are divided into two families. Each line of the first family does not intersect any line of the same family, but it does intersect any line of the other family and viceversa.

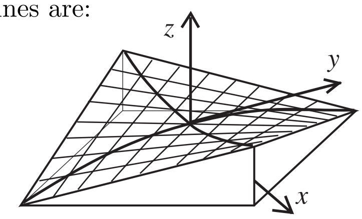


The two family of lines can be written by using an algebraic trick:

$$z \cdot 1 = \left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) \quad \longrightarrow \quad z \Big/ \left(\frac{x}{a} - \frac{y}{b}\right) = \left(\frac{x}{a} + \frac{z}{c}\right) = k$$

You can exchange the way of forming the fractions and then the families of lines are:

$$\begin{cases} \left(\frac{x}{a} - \frac{z}{c}\right) = k\left(1 - \frac{y}{b}\right) \\ \left(1 + \frac{y}{b}\right) = k\left(\frac{x}{a} + \frac{y}{b}\right) \end{cases} \quad \begin{cases} \left(\frac{x}{a} - \frac{y}{b}\right) = k\left(1 + \frac{y}{b}\right) \\ \left(1 - \frac{y}{b}\right) = k\left(\frac{x}{a} + \frac{y}{b}\right) \end{cases} \quad (k \in \mathbb{R})$$



This quadric has no symmetry center, one symmetry axis and two symmetry planes. It can never be a revolution surface, since there are not ellipses on it.

Intersection between a quadric and a plane

The intersection between a quadric and a plane is always a conic.
Almost all the points on a quadric have a tangent plane, that is a plane containing all the lines tangent to the quadric in that point. The only exceptions are the vertex on a cone and some points of cylinders of degenerated conics.

Tangent planes to a quadric

If $f(x, y, z) = 0$ is the equation of the quadric $\mathcal{Q}$ and $P(x_0, y_0, z_0)$ is a point on $\mathcal{Q}$, then the tangent plane to $\mathcal{Q}$ in P is $a(x - x_0) + b(y - y_0) + c(z - z_0)$, where $\vec{v} = (a, b, c)$ is the gradient of f in P, that is the vector $\left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}\right)$ calculated in P.

The intersection between a quadric and a tangent plane is always a degenerated conic. There are three cases:
If the intersection consists of two coincident lines, we say that the point is parabolic.
If the intersection consists of one point (degenerated ellipse), we say that the point is elliptic.
If the intersection consists of two intersecting lines (degenerated hyperbola), we say that the point is hyperbolic.
All the points on the same quadric are of the same kind (i.e. parabolic or elliptic or hyperbolic).

Cylinders and cones have parabolic points.
Ellipsoids, hyperboloids of two sheets ad elliptic paraboloids have elliptic points.
Hyperboloids of one sheet ad hyperbolic paraboloids have hyperbolic points.

Some criteria to recognize quadrics

As already said, the 4×4 matrix A tells how degenerated is the quadric. Namely:

If rank $(A) = 1$ then the quadric consists of two coincident planes.
If rank $(A) = 2$ then the quadric consists of two planes (possibly non real).
If rank $(A) = 3$ then the quadric is a cylinder or a cone.
If rank $(A) = 4$ then the quadric is an ellipsoid or a paraboloid or an hyperboloid.
The 3×3 quadric B is the matrix associated to the quadratic form of the equation of the quadric. It can be proved the signature of the quadratic form does not change if the quadric is rotated or translated, but can change the sign (i.e. the signatures $(+, +, -)$ and $(+, -, -)$ are equivalent and so on)
We summarize the signatures of the quadratic form of the main quadrics:

Parabolic cylinder	$(+,0,0)$	Elliptic cylinder	$(+,+,0)$
Hyperbolic cylinder	$(+,-,0)$	Non-real cone	$(+,+,+)$
Real cone	$(+,+,-)$	Ellipsoid (real or not)	$(+,+,+)$
Hyperboloid (1 or 2 sheets)	$(+,+,-)$	Elliptic paraboloid	$(+,+,0)$
Hyperbolic paraboloid	$(+,-,0)$		

In order to distinguish between real or not-real ellipsoid and hyperboloid of 1 or 2 sheets, the following criteria can be used:

If $\det(A) > 0$ the quadric is a non-real ellipsoid.
If $\det(A) > 0$ the quadric is a hyperboloid of 1 sheet.

Center and symmetry axes of a quadric

If the quadric has symmetry center, the center can be found by solving the linear system of partial derivatives of the equation with respect to x, y and z: $\left\{\dfrac{\partial f}{\partial x} = 0 \quad ; \quad \dfrac{\partial f}{\partial y} = 0 \quad ; \quad \dfrac{\partial f}{\partial z} = 0\right\}$

To find symmetry axes, one needs to find the eigenvectors of the matrix A.
If the quadric has symmetry center C, then the lines passing through C and having the directions of the eigenvectors are the axes.

Note that if the quadric is a revolution surface, then one of the eigenvalue has multiplicity 2 and so there are ∞^2 symmetry axes corresponding to that eigenvalue.
The cone and the hyperboloids have a main axis (in the canonical form, the z axis). This axis correspond to the "different" eigenvalue (the one whose sign is different).
If the quadric is a paraboloid, then it has no center. The direction of the symmetry axis is given by the eigenvector relative to the eigenvalue 0.
The vertex can be found by intersecting the quadric with all the planes which are orthogonal to the symmetry axis. There is only one plane whose intersection with the paraboloid is a degenerated conic. The center of this conic is the vertex.