Orthogonal matrices

01. Replace the stars with number in the third column of the matrix P so that P is orthogonal and has determinant 1. In how many ways is it possible to do it?

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & * \\ -1/\sqrt{2} & 1/\sqrt{6} & * \\ 0 & 2/\sqrt{6} & * \end{pmatrix}$$

02. Replace the stars with numbers in the matrix $Q \in M_{33}(\mathbb{R})$, so that Q is orthogonal. Do it *in all possible ways.*

$$Q = \begin{pmatrix} -2/3 & 2/3 & * \\ 2/3 & 1/3 & * \\ * & * & * \end{pmatrix}$$

Spaces with a scalar product

11. Let $W = L\{(1,1,0),(0,1,1)\}$ be a subspace of $\mathbb{R}^3$ equipped with the euclidean norm.
    a. Find an orthonormal basis for W and calculate the projection p of the vector $v = (1,2,0)$ onto W.
    b. Verify that $v - p$ is orthogonal to W.
12. In the vector space $\mathbb{R}^4$ equipped with euclidean norm, find the distance between the vector $v = (1,1,1,1)$ and the subspace $W = L\{(0,1,0,0),(0,0,1,2),(1,1,1,0)\}$.
13. Explain why the matrix A induces a scalar product in $\mathbb{R}^2$ and find an orthonormal basis of $\mathbb{R}^2$ equipped with this scalar product.

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

14. Consider the matrix $A \in M_{33}(\mathbb{R})$.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & k \\ 0 & k & 1 \end{pmatrix}$$

    a. Find all $k \in \mathbb{R}$ for which the following product $\langle v, w\rangle_* = v^T \cdot A \cdot w$ in the vector space $\mathbb{R}^3$ (v e w column vectors), is a scalar product.
    b. Choose a k such that $(1,0,1)$ is orthogonal to $(1,1,0)$ or to $(0,1,0)$ or to $(-1,1,0)$ using the previous scalar product.
    c. Choose the k found above and calculate an o.n. basis for $\mathbb{R}^3$.
15. In the vector space $C^\infty[1,2]$ with the usual scalar product (by the integral), verify the Cauchy-Schwarz inequality for the two functions $f_1 = \dfrac{ax^2+b}{x}$ and $f_2 = \dfrac{1}{x}$ for all $a, b \in \mathbb{R}$. Find the couples a, b which make the inequality an equality.
16. Let $V = L\{1, x\}, V_1 = L\{1, x, x^2\}$ be subspaces of $C^\infty([1,1])$ (usual scalar product).
    a. Find an orthonormal basis of V and one of V_1 by means of Gram-Schmidt process.
    b. Find the projection of the function $f(x) = x^2$ onto V.
    Which is the relation between $f(x)$ and its projection $p(x)$?
    What does the minimum property mean in this case? Calculate this minimum.
17. In the vector space $C^\infty[1,2]$ (usual scalar product).
    a. Find an orthonormal basis for the subspace $W = L\{1/x, x\}$.
    b. Calculate the projection $p(x)$ of the function $f(x) = x^2$ onto W.
    c. Let $f(x) \in C^\infty[1,2]$ be a function and $p(x)$ be its projection onto W, Prove that, if at least one between $\int_1^2 f(x)x\,dx$ and $\int_1^2 \frac{f(x)}{x}\,dx$ is not zero, then $\int_1^2 f(x)p(x)\,dx > 0$.
18. In the $\mathbb{R}$-vector space $C^0([-1,1])$ define a product in the following way: $\langle f, g\rangle_1 = \int_{-1}^{1} f \cdot g \cdot x^2\,dx$
    a. Say why $\langle , \rangle_1$ is a scalar product.
    b. Find an o.n. basis for the subspace $W = L\{1, x\}$.
19. Find $a \in \mathbb{R}$ $(a > 0)$ such that in the space $C^0[0,a]$ $(a > 0)$ (usual scalar product) the two vectors $f_1 = 2 - x$ $f_2 = x$ are orthogonal.
    For this a, find the projection of the function $g = x^2$ onto the subspace $L\{f_1, f_2\}$

Matrix norms and condition number

21. Given the symmetric matrix $A = \begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix}$ calculate $\| A \|_2$ and $\text{cond}_2(A)$

22. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ Calculate:
$$\begin{array}{ccc} \| A \|_2 & \| A \|_1 & \| A \|_\infty \\ \text{cond}_2(A) & \text{cond}_1(A) & \text{cond}_\infty(A) \end{array}$$

23. Let $A = \begin{pmatrix} 0 & 3 & 2 \\ 3 & -1 & 3 \\ 2 & 3 & 0 \end{pmatrix}$
    a. Knowing that $P_A(x) = (x+4)(x+2)(5-x)$, calculate $\| A \|_2, \| A \|_1$ and $\| A \|_\infty$.
    b. Calculate $\text{cond}_2(A)$
    c. Let $b = [1 \ , \ 1 \ , \ 1]^T$ and $b_1 = [1 \ , \ 1 \ , \ 2]^T$ and let x and x_1 be the solutions of linear systems $Ax = b$ e $Ax = b_1$. Estimate an upper bound for $\dfrac{\| x - x_1 \|_2}{\| x \|_2}$ without calculating x and x_1.

24. Let $A = \begin{pmatrix} -1 & 2 & -1 \\ 2 & 2 & -2 \\ -1 & -2 & -1 \end{pmatrix}$
    a. Calculate the eigenvalues of A.
    b. Calculate $\| A \|_2$ and $\text{cond}_2(A)$.
    c. Find among the matrices $A+I$, $A-I$, $A-2I$ the one with with the best condition number and the one with with the worst condition number.

25. We provide some data about the matrix A:
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ -1 & 2 & 4 & 5 \\ -1 & -1 & 4 & 4 \\ 3 & 3 & 2 & 3 \end{pmatrix}$$
    - The eigenvalues of the matrix $A^T \cdot A$ are (approximately) 0.0022 , 2.4175 , 18.8818 , 91.6986
    - The solution of the system $Au = (1 \ , \ 4 \ , \ 4 \ , \ 4)^T$ is $x = (2/3 \ , \ -2/3 \ , \ -1 \ , \ 2)^T$

    Using the given data, give an estimate for the solution x_1 of $Au = (1.1 \ , \ 4 \ , \ 3.9 \ , \ 4)^T$

26. Find all the $k \in \mathbb{R}$ for which it is possible to calculate the condition number of A_k and draw the graphic of the function $f(k) = \text{cond}_2(A_k)$ (defined for all k found above)
$$A_k = \begin{pmatrix} 2 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & k \end{pmatrix}$$

27. Replace the stars with numbers in the matrix A, so that A is symmetric and orthogonal.
$$A = \begin{pmatrix} 6/7 & * & * \\ 2/7 & 3/7 & * \\ * & * & * \end{pmatrix}$$
    a. Using the properties of A, calculate $\text{cond}_2(A)$ and find its eigenvalues.
    b. Calculate $\text{cond}_2(A - kI)$ for all k for which it make sense.