# On Numerical Semigroups and the Order Bound

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[1] **Abstract.** Let $S = \{ s_0 = 0 < s_1 < ... < s_i ... \} \subseteq \mathbb{N}$ be a numerical non-ordinary semigroup; then set, for each $i$, $\nu_i := \# \{(s_i - s_j, s_j) \in S^2\}$. We find a non-negative integer $m$ such that $d_{ORD}(i) = \nu_{i+1}$ for $i \geq m$, where $d_{ORD}(i)$ denotes the order bound on the minimum distance of an algebraic geometry code associated to $S$. In several cases (including the acute ones, that have previously come up in the literature) we show that this integer $m$ is the smallest one with the above property. Furthermore it is shown that every semigroup generated by an arithmetic sequence or generated by three elements is acute. For these semigroups, it is also found the value of $m$.

*Index Therms.* Numerical semigroup, Weierstrass semigroup, semigroup generated by an arithmetic sequence, algebraic geometry code, order bound on the minimum distance.

## 1 Introduction

Let $\mathbb{N}$ denote the set of all non-negative integers and let $S \subseteq \mathbb{N}$ be a numerical semigroup, $S = \{s_0 = 0 < s_1 < ... < s_i < ...\}$. The associated sequence $\{\nu_i\}_{i \in \mathbb{N}}$ is defined by

$$\nu_i := \#\{(s_j, s_k) \in S^2 \mid s_j + s_k = s_i\}.$$

When $S$ is the *Weierstrass semigroup* of a family $\{\mathcal{C}_i\}_{i \in \mathbb{N}}$ of one-point algebraic geometry ($AG$) codes (see, e.g. [6]), Feng and Rao proved that the minimum distance of the code $\mathcal{C}_i$ can be bounded by the so called *order bound* [4] defined by means of the sequence $\{\nu_i\}_{i \in \mathbb{N}}$:

$$d_{ORD}(C_i) := min\{\nu_j \ : \ j > i\}.$$

The sequence $\{\nu_i\}_{i \in \mathbb{N}}$ is not decreasing from a certain $i$ [9]; then there exists an integer $m$ determining the largest point at which the sequence decreases, that is, $d_{ORD}(C_i) = \nu_{i+i}$ for $i \geq m$. The parameter $m$ is already known for the so-called *acute* semigroups [1] and in such a case it is equal to

$$m = min\{c + c' - 2 - g, \ 2d - g\}$$

where $c, c', d$ are the conductor, the subconductor and the dominant of the semigroup, as in (2.1).

In this work we develope an analysis of $m$ and we classify semigroups in terms of a new parameter $t$. Using this classification and some small condition on the dominant $d$, we derive new results for the parameter $m$. In several cases the actual value of $m$ is stated and in the remaining cases an upper bound $m' \geq m$ is given (Theorem 3.1). In particular, $d_{ORD}(C_i) = \nu_{i+i}$ for all $i \geq m'$ (3.2).
A consequence of these results is that for any numerical semigroup

$$m \leq min\{c + c' - 2 - g, \ 2d - g\}$$

and that $m = min\{c + c' - 2 - g, \ 2d - g\}$ if and only if either $c + c' - 2 \leq 2d$ or $t = 0$ (Corollary 3.3).

Further, in Section 4, we study the classes of acute semigroups, semigroups generated by an arithmetic sequence and semigroups generated by an almost arithmetic sequence. We show that

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semigroups generated by an arithmetic sequence are acute as well as the semigroups generated by three integers, which are a particular case of semigroups generated by an almost arithmetic sequence.

Finally an analysis on the semigroups of Cohen-Macaulay type 2 or 3 shows that all numerical semigroups of type 2 are acute while semigroups of type 3 are acute except for the case when the subdominant and the subconductor satisfy $d' \leq c' - 3$. Furthermore, for all numerical semigroups generated by an arithmetic sequence and for all numerical semigroups of type 2 or 3, a formula for the parameter $m$ is presented.

In the next section (Section 2) we fix the setting and notation of the paper, moreover we recall some known results for the convenience of the reader.

## 2 Preliminaries

We begin by giving the setting of the paper.

**Setting 2.1** In all the article we shall use the following notation. Let $\mathbb{N}$ denote the set of all non-negative integers. A *numerical semigroup* is a subset $S$ of $\mathbb{N}$ containing 0, closed under summation and with finite complement in $\mathbb{N}$.
We denote the elements of $S$ by

$$s_0 = 0 \ < s_1 := e \ < s_2 \ < \ldots < s_i \ \ldots \quad \text{for every} \quad i \in \mathbb{N}$$

and we set $S_+ := S \setminus \{0\}$, $S(1) := \{b \in \mathbb{N} \mid \ b + S_+ \subseteq S\}$.

The following is a list of symbols and relations associated to a semigroup $S$, to be used in the sequel.

$$\begin{array}{rcl}
H & := & \mathbb{N} \setminus S, \text{ the set of gaps of } S \\
g & := & \#(\mathbb{N} \setminus S), \text{ the number of gaps of } S \\
g(i) & := & \#\{\sigma \in H \mid \sigma < s_i\}, \text{the number of gaps of S which are smaller than } s_i, \text{ for } i \in \mathbb{N} \\
c & := & min\ \{r \in S \mid r + \mathbb{N} \subseteq S\} \text{ is the } \textit{conductor} \text{ of } S \\
n & := & c - g, \text{ the number of the elements of S preceding the conductor, so that } c = s_n \\
d & := & s_{n-1} \text{ the greatest element in } S \text{ preceding } c, \text{ is } \textit{the dominant} \text{ of } S \\
\ell & := & c - 1 - d = \#\{\sigma \in H \mid \sigma > d\}, \text{ the number of gaps of S greater than } d \\
c' & := & s_p = max\{s_i \in S \mid s_i \leq d \ \ and \ \ s_i - 1 \notin S\} \text{ is the } \textit{subconductor} \text{ of S} \\
d' & := & s_{p-1}, \text{ the greatest element in S preceding } c', \text{ when } d > 0 \\
\tau & := & \#(S(1) \setminus S), \text{ is the } Cohen-Macaulay\ type \text{ of } S\ (CM-type \text{ for brevity}) \\
e & := & s_1 \text{ is the } \textit{multiplicity.} \\
H_1 & := & \{\sigma \in H \mid c - 1 - \sigma \in S\}, \text{ the set of gaps of the } first\ type \text{ of } S \\
H_2 & := & \{\sigma \in H \mid c - 1 - \sigma \notin S\}, \text{ the set of gaps of the } \textit{second type} \text{ of } S \\
\mu & := & max\{\sigma \in H_2\}, \text{ the greatest gap of the second kind of } S, \text{ when } H_2 \neq \emptyset
\end{array}$$

For $s_i \in S$, following [6] and [1], we shall denote according to the convenience by

$$\begin{array}{lcl}
N_i \text{ or } N(s_i) & := & \{(s_j, s_k) \in S^2 \mid s_i = s_j + s_k\} = \{(s_j,\ s_i - s_j) \in S^2\} \\
A_s(s_i) & := & \{(x, y) \in N_i \mid x \geq s\}, \text{ for every } s \in S \\
D_i \text{ or } D(s_i) & := & \{(x, y) \in H^2 \mid x + y = s_i\} = \{(x, s_i - x) \in H^2\} \\
\nu_i \text{ or } \nu(s_i) & := & \#N_i, \text{ the cardinality of } N_i \\
d_{ORD}(i) & := & min\{\nu_j \mid j > i\}, \text{ the order bound.}
\end{array}$$

We shall always assume that $e > 1$, so that $S \neq \mathbb{N}$. With this notation the semigroup has the following shape (where " $*$ " denotes gaps and " $\longleftrightarrow$ " intervals without any gap):

$$S = \{0,\ * \ldots *, e,\ \ \ldots\ \ d',\ * \ldots *, , c' \longleftrightarrow d,\ \overbrace{* \ldots *}^{\ell \text{ gaps}},\ c \rightarrow\}.$$

Recall also that a semigroup $S$ is called

- *ordinary* if $S = \{0\} \cup \{i \in \mathbb{N},\ i \geq c\}$,

- *acute* if $S$ is ordinary or if $S$ is non-ordinary and $c, d, c', d'$ satisfy $c - d \leq c' - d'$. [1, Defs. 5.1 and 5.6.]
- *symmetric* if for every $x \in \mathbb{N},\ x \in S \Longleftrightarrow c - 1 - x \notin S$, equivalently, $H_2$ is empty. Also, $S$ is symmetric if and only if the Cohen-Macaulay type of $S$ is one.

The aim of this paper is the study of the behaviour of integers $\nu_i$'s: in the next theorem we collect some important well-known results on these parameters.It is well known that the sequence $(\nu_i)_{i\in\mathbb{N}}$ is not decreasing for $i >> 0$. In [1] the author finds the smallest integer $m$ such that $d_{ORD}(i) = \nu_{i+1}$ for all $i \geq m$, when the Weierstrass semigroup $S$ is *acute* (see (2.4) and (2.5) below). Moreover $m = 0$ if and only if $S$ is ordinary. In fact, the only numerical semigroups for which the sequence $(\nu_i)_{i\in\mathbb{N}}$ is non-decreasing are the ordinary semigroups [1, Th. 7.3].

**Theorem 2.2** *Let $S$ be as in 2.1, and let $i \in \mathbb{N}$. Then*

(1) $\nu_i = i - g(i) + \#D_i + 1, \quad \forall\ i \in \mathbb{N}.$ [9, Th. 3.8]

(2) $\nu_i = i + 1 - g,$ *for every* $i$ *such that* $s_i \geq 2c - 1$.

As a consequence of Theorem 2.2, the sequence $(\nu_i)_{i\in\mathbb{N}}$ is non-decreasing for $i$ large enough.

**Definition 2.3** *We define the parameters* $m$ *and* $t$ *as follows*

$$m := min\{j \in \mathbb{N} \text{ such that the sequence } (\nu_i)_{i\in\mathbb{N}} \text{ is non-decreasing for } i > j\}$$

$$t := min\{j \in \mathbb{N} \text{ such that } d - t \in S \text{ and } d - \ell - t \notin S\}.$$

**Remark 2.4** Theorem 2.2 implies that $m > 0$ for every non-ordinary semigroup, and that $m \leq 2c - 2 - g$, namely $s_m \leq 2c - 2$. Recalling the definition of $d_{ORD}(i)$ above, one has:

- $d_{ORD}(i) = \nu_{i+1}, \quad for \ \ every \ \ i \geq m.$

It then becomes important to find the integer $m$, and, for this, to study the behavior of the sequence $(\nu_i)_{i\in\mathbb{N}}$. Clearly it is enough to consider the cases: $s_i \leq 2c - 2$, namely $i \leq 2c - g - 2$.
The meaning of $t$ will be clear in the following sections.

We recall next theorem which gives $m$ for acute semigroups [1, Th. 6.3].

**Theorem 2.5** *Let $S$ be a non-ordinary acute semigroup. Then,*

$$m = min\ \{c + c' - 2 - g,\ 2d - g\}.$$

# 3 The order bound on the minimum distance

By studying the behaviour of the $\nu_i$'s, we can find the integer $m$ defined in (2.3) for several classes of semigroups, which properly include the acute ones. In the other cases it is possible to give upper bounds for $m$.

The aim of this section is to prove the following Theorem 3.1 and Theorem 3.2.

**Theorem 3.1** *With Setting* 2.1, *let $S$ be a non-ordinary semigroup. Let $m$, $t$ be as in* (2.3). *Then,*

(1) *When* $c + c' - 2 \leq 2d$: $\quad m = c + c' - 2 - g.$

(2) *When* $c + c' - 2 > 2d$:

  (a) *If* $0 \leq t \leq 2$: $\quad m = 2d - g - t.$

  (b) *If* $t = 3$: $\quad m = 2d - g - 3$ *if and only if* $\{d-1, d-2\} \cap S \neq \{d-2\}$.
  $m \leq 2d - g - 4$ *if* $\{d-1, d-2\} \cap S = \{d-2\}$.

(c) *If* $t = 4$: $m = 2d - g - 4$ *if and only if* $\{d-1, d-2, d-3\} \cap S \neq \{d-3\}$.
$m \leq 2d - g - 5$ *if* $\{d-1, d-2, d-3\} \cap S = \{d-3\}$.

(d) *If* $t \geq 5$: $m \leq 2d - g - 4$.

**Theorem 3.2** *Let $S$ be a non-ordinary semigroup associated to a family of AG codes $\mathcal{C}_i$, $i \in \mathbb{N}$. Then the equality* $d_{ORD}(i) = \nu_{i+1}$ *holds in the following cases.*

(1) *When* $c + c' - 2 \leq 2d$: *if and only if* $i \geq c + c' - 2 - g$.

(2) *When* $c + c' - 2 > 2d$:

(a) *when* $0 \leq t \leq 2$: *if and only if* $i \geq 2d - g - t$

(b) *when* $t = 3$ *and* $\{d-1, d-2\} \cap S \neq \{d-2\}$: *if and only if* $i \geq 2d - g - 3$
*when* $t = 3$ *and* $\{d-1, d-2\} \cap S = \{d-2\}$: *for each* $i \geq 2d - g - 4$

(c) *when* $t = 4$ *and* $\{d-1, d-2, d-3\} \cap S \neq \{d-3\}$: *if and only if* $i \geq 2d - g - 4$
*when* $t = 4$ *and* $\{d-1, d-2, d-3\} \cap S = \{d-3\}$: *for each* $i \geq 2d - g - 5$

(d) *when* $t \geq 5$: *for every* $i \geq 2d - g - 4$. *More precisely:*

- *when* $\left[\begin{array}{ll} \text{either} & d-4 \in S,\ d-\ell-4 \in S, \\ \text{or} & d-4 \notin S,\ d-\ell-4 \notin S \end{array}\right.$ *and* $\{d-1, d-2, d-3\} \cap S = \{d-2\}$:
*if and only if* $i \geq 2d - g - 4$.
- *for each* $i \geq 2d - g - 5$ *in the other cases.*

Before proving the theorems we derive the following consequences.

**Corollary 3.3** (1) $m \leq min\{c + c' - 2 - g,\ 2d - g\}$, *for every non-ordinary semigroup S.*

(2) $m = min\{c + c' - 2 - g,\ 2d - g\}$ *if and only if either* $c + c' - 2 \leq 2d$, *or* $t = 0$.

Proof. (2) Let $m = min\{c + c' - 2 - g,\ 2d - g\}$: if $c + c' - 2 > 2d$, then $m = 2d - g$, and so $t = 0$, by (3.1.2.a). The other implication follows by (3.1.1) and (3.1.2.a). $\diamond$

**Proposition 3.4** *Every non-ordinary acute semigroup satisfies either* $c + c' - 2 \leq 2d$ *or* $t = 0$.

Proof. In fact if $c + c' - 2 \geq 2d + 1$, and $S$ is acute, one has $c' - 1 > d - \ell$ and $c' - d' \geq \ell + 1$, so $d - \ell \geq d - c' + d' + 1 \geq d' + 1$, so $d' + 1 \leq d - \ell < c' - 1$, then $d - \ell \notin S$, and $t$=0. $\diamond$

**Remark 3.5** (1) Let $S$ be a non-ordinary acute semigroup, then the equality $m = min\{c + c' - 2 - g,\ 2d - g\}$ [1, Th. 6.3] follows from (3.4) and (3.3.2).

(2) Not all numerical semigroups satisfying either $c + c' - 2 \leq 2d$, or $t = 0$ are acute. A counterexample is given by the semigroup $S =< 8, 21, 36, 51, 62 >$ considered in (3.14.B).

(3) Notice that $c + c' - 2 < 2d = c + d - \ell - 1 \iff c' - 1 < d - \ell \iff c' \leq d - \ell < d \Longrightarrow d - \ell \in S$, that is, $t > 0$. Then, $d - \ell \notin S \Longrightarrow c + c' - 2 \geq 2d$. Further if $c + c' - 2 = 2d$ one has $d - \ell = c' - 1 \notin S$. It follows (see the proof of (3.4)):

*If $S$ is acute the condition* $c + c' - 2 \geq 2d$ *is equivalent to* $d - \ell \notin S$.

## 3.1 Proof of theorems 3.1 and 3.2.

In order to prove Theorem 3.1 and Theorem 3.2 we need some preliminary results.

**Lemma 3.6** *With Setting 2.1:*

(1) *If* $c - 2 \in S$, *then the semigroup S is acute and* $m = c + c' - 2 - g$.

(2) *If S is a symmetric semigroup, then,*

(a) $d = c - 2$, *and so* $S$ *is acute.*

(b) $c' = c - e$.

(c) *If* $S$ *is non-ordinary :* $d' = c' - 2 \Longleftrightarrow e + 1 \notin S$.

(d) $c + c' - 2 \leq 2d$.

(3) *If* $\alpha \in \mathbb{N}$ *such that* $1 < \alpha \leq e$, *then* $c - \alpha \notin H_1$.

(4) *If* $c - 2 \notin S$, *then* $c - 2 \in H_2$.

(5) *If* $\mu < c - 2$, *then* $c - 2 \in S$.

(6) $1 \leq \ell \leq e - 1$.

Proof. The proof is straightforward. $\diamond$

**Lemma 3.7** *Let* $s_i \in S$, $s_i \leq 2c-1$. *For each* $s \in S$ *consider the set* $A_s(s_i) = \{(x, y) \in N_i \mid x \geq s\}$ *as in (2.1). Then,*

(1) $A_0(s_i) = N_i$.

(2) $x \neq y$, *for every* $(x, y) \in A_c(s_i)$.

(3) $\#A_c(s_i) = \#\{y \in S \mid y \leq s_i - c\}$.

(4) $A_{c+1}(s_{i+1}) = \{(x + 1, y) \mid (x, y) \in A_c(s_i)\}$.

(5) $A_c(s_{i+1}) = \begin{cases} A_{c+1}(s_{i+1}), & if \quad s_{i+1} - c \notin S \\ A_{c+1}(s_{i+1}) \cup \{(c, s_{i+1} - c)\}, & if \quad s_{i+1} - c \in S. \end{cases}$

(6) $\#A_c(s_i) = \#A_{c+1}(s_{i+1})$.

(7) $\#A_c(s_{i+1}) = \begin{cases} \#A_c(s_i) & if \quad s_{i+1} - c \notin S \\ \#A_c(s_i) + 1 & if \quad s_{i+1} - c \in S. \end{cases}$

(8) *If* $d - \ell - k \in S$, *then* $(c, d - \ell - k) \in A_c(2d - k + 1)$.

Proof. (1), (2), (3), (5) are immediate.
(4) If $(x, y) \in A_c(s_i)$ then $s_i \geq c$: hence $s_{i+1} = s_i + 1$. Therefore we can define a correspondence

$$\phi : A_c(s_i) \longrightarrow A_{c+1}(s_{i+1}) \text{ by}$$
$$(x, y) \mapsto (x + 1, y).$$

This map $\phi$ is clearly one to one, hence (4) follows.
(6) and (7) follow directly from (4) and (5).
(8) Write $2d - k = d + c - \ell - 1 - k$, then $2d + 1 - k = c + (d - \ell - k)$. This proves (8). $\diamond$

By using the above lemma we can easily evaluate the set $N(s_i)$ for every $s_i \geq 2d + 1$.

**Lemma 3.8** *Let* $s_i \geq 2d + 1$. *Then,*

$$N(s_i) = \{(x, y) \in S^2 \mid \textit{either } (x, y) \in A_c(s_i), \textit{ or } (y, x) \in A_c(s_i)\}.$$

Proof. Since $s_i \geq 2d + 1$, the equality $x + y = s_i$, with $x < c$, (hence $x \leq d$, by (2.1)) yields to $y = s_i - x \geq 2d + 1 - d = d + 1$. Therefore $y \geq c$. $\diamond$

**Proposition 3.9** *Assume that* $S$ *is non-ordinary. Then,*

(1) (a) $\nu_{i+1} \geq \nu_i$, *for every* $i \geq 2d + 1 - g$, *equivalently, for every* $s_i \geq 2d + 1$.

(b) $\nu_i = 2c - 2g$, *for* $c + d - g \leq i \leq 2c - g - 1$, *equivalently, for* $c + d \leq s_i \leq 2c - 1$.

(2) *If* $i = 2d - g$, *i.e.* $s_i = 2d$, *then* $\nu_{i+1} = \left[ \begin{array}{lll} \nu_i + 1 & if & d - \ell \in S \\ \nu_i - 1 & if & d - \ell \notin S. \end{array} \right.$

Proof. Part $(1.a)$ follows from (3.8) and (3.7.7).
$(1.b)$ When $2d+1 \leq s_i \leq 2c-1$, by (3.7.2) and by (3.8) one has $\nu_i = \#N_i = 2\#A_c(s_i)$. In particular, when $c + d \leq s_i \leq 2c - 1$, we have $\#A_c(s_i) = c - g$ by (3.7.3).
(2) Let now $s_i = 2d$. Clearly, $\{(d, d)\} \in N_i$. If $(x, y) \in N_i$, $(x, y) \neq (d, d)$, the same argument as in the proof of (3.8) shows that either $(x, y) \in A_c(s_i)$ or $(y, x) \in A_c(s_i)$. Then by (3.7.2), (3.7.7) one has $\nu_i = 1 + 2\#A_c(s_i)$ and
either $\nu(2d + 1) = 2\#A_c(s_i) + 2$, if $d - \ell \in S$,
or $\nu(2d + 1) = 2\#A_c(s_i)$, if $d - \ell \notin S$. This proves statement (2) . ⋄

Since $t = 0$ if and only if $d - \ell \notin S$, we obtain the following corollary.

**Corollary 3.10** $m = 2d - g$ *if and only if* $t = 0$.

**Remark 3.11** If $t = 0$ then necessarily $2d - g \leq c + c' - 2 - g$ (3.5.3 ) and so the same formula for acute semigroups (Theorem 2.5) still applies for semigroups with parameter $t = 0$. This result is proved independently in [7, Th.3.11].

To investigate the remaining cases, we use different techniques according to $c + c' - 1 \leq 2d + 1$, or $c + c' - 1 > 2d + 1$.

**Lemma 3.12** *Assume that* $S$ *is non-ordinary and that* $c + c' - 1 \leq 2d + 1$ . *Then,*

(1) $D(s_i) = \emptyset$ *for every* $s_i \in S$ *such that* $c + c' - 1 \leq s_i \leq 2d + 1$.

(2) $D(c + c' - 2) = \{(c' - 1,\ c - 1), (c - 1,\ c' - 1)\}$.

(3) *Let* $c + c' - 2 = s_j$. *Then* $\nu(c + c' - 2) = \nu_j = j + 1 - g + \#D_j = j + 3 - g.$
$\nu(c + c' - 1) = \nu_{j+1} = (j + 1) + 1 - g = j + 2 - g.$

Proof. First note that $d \geq 2$, since $S$ is non-ordinary and $1 \notin S$. Hence also $c' \geq 2$ and $c + c' - 2 \geq c$.
(1) By the above observation, if $c + c' - 1 \leq s_i \leq 2d + 1$, one has
$$s_i = c + k, \quad \text{with} \quad c' - 1 \leq k \leq d - \ell.$$
Also, $c' \leq k + 1 \leq d - \ell + 1 \leq d \Longrightarrow k + 1 \in S$. Let now $(x, y) \in D_i$, so that $x + y = s_i$ and $(x, y) \in H^2$. Note that $x, y \leq c - 2$; indeed if $x = c - 1$, then $y = s_i - c + 1 = k + 1 \in S$. Thus
$y = s_i - x \geq c + c' - 1 - (c - 2) = c' + 1$.
Therefore $d < y < c$ i.e., $y = d + q$, with $1 \leq q \leq \ell$ and so
$x = s_i - d - q = c + k - d - q = \ell + k + 1 - q$, where $0 \leq \ell - q \leq \ell - 1$, $c' \leq k + 1 \leq d - \ell + 1$.
Then one obtains $c' \leq x \leq d$: contradiction, since $x \notin S$. This proves (1).
(2) Let $s_i = c + c' - 2$, and let, as above, $(x, y) \in D_i$. Then $x, y \geq c' - 1$. Indeed $x < c' - 1$ would imply $y > c + c' - 2 - c' + 1 = c - 1$ and so $y \in S$. Further if $x > c' - 1$, then $x \geq d + 1$. So
$y = c + c' - 2 - x \leq c + c' - 2 - d - 1 \leq 2d - d - 1 = d - 1$, hence $y \leq c' - 1$.
These arguments show that either $y = c' - 1$ or $x = c' - 1$ and we are done.
(3) Let $s_j = c + c' - 2$: by Theorem 2.2 and Lemma 3.12.2 one gets:
$\nu(c + c' - 2) = \nu_j = j + 1 - g + \#D_j = j + 3 - g.$
$\nu(c + c' - 1) = \nu_{j+1} = (j + 1) + 1 - g = j + 2 - g.$ ⋄

**Proposition 3.13** *Assume that* $S$ *is non-ordinary and that* $c + c' - 1 \leq 2d + 1$. *Then,*

$$m = c + c' - 2 - g \qquad \text{and} \qquad \nu_{m+1} = \nu_m - 1.$$

Proof. Let $s_j = c + c' - 2$.
Case A: $c+c'-1 = 2d+1$. From the equalities in (3.12.3) and from (3.9.1.a), we deduce that $m = j$.
Case B: $c + c' - 1 < 2d + 1$. Again by (3.12.2) and (2.2.1) one gets:

$\nu(c + c' + h) = \nu_{j+2+h} = j + 2 + h + 1 - g = j + 3 - g + h,$ for every $h \in [0, 2d + 1 - c - c']$

and we are done by (3.9.1.a), and by $(i)$, $(ii)$ above. $\diamond$

**Example 3.14** We show two examples for the cases (A), (B) in the above proof.

Case (A): let $S = \{0, 6, 8 \rightarrow\}$, generated by $< 6, 8, 9, 10, 11, 13 >$. $c = 8,\ d = c' = 6,\ d' = 0,\ g = 6,$ $\ell = 1,\ d - \ell = d - 1 \notin S,\ c + c' - 1 = 13 = 2d + 1$. This semigroup is acute.
By (3.13), $s_m = c + c' - 2 = 2d = 12$. The sequence $\{\nu_i\}_{i\in\mathbb{N}}$ is:

$$\begin{bmatrix} 0 & 6 & 8 & ... & 11 & 12 = s_m & 13 & 14 & 15 = 2c-1 & 16 = 2c & \rightarrow \\ \nu_0 & \nu_1 & \nu_2 & ... & \nu_5 & \nu_6 = \nu_m & \nu_7 & \nu_8 & \nu_9 & \nu_{10} & \rightarrow \\ 1 & 2 & 2 & ... & 2 & 3 & 2 & 4 & 4 & 5 & \rightarrow \end{bmatrix}$$

Case (B): let $S = \{0, 8, 16, 21, 24, 29, 32, 36, 37, 40, 42, 44, 45, 48, 50, 51, 52, 53, 56 \rightarrow\}$, generated by $< 8, 21, 36, 51, 62 >$. $c = 56,\ d = 53,\ c' = 50,\ d' = 48,\ g = 38,\ \ell = 2,\ d - \ell = 51 \in S,$ $c + c' - 1 = 105 < 107 = 2d + 1$. This semigroup is not acute since $c' - d' = 2 < c - d = 3$, and $m = c + c' - 2 - g = 66$. In fact the sequence $\{\nu_i\}_{i\in\mathbb{N}}$ is:

$$\begin{bmatrix} 0 & ... & 104 = s_m & 105 & 106 = 2d & 107 & 108 & 109 & 110 & 111 = 2c-1 & 2c & \rightarrow \\ \nu_0 & ... & \nu_{66} & \nu_{67} & \nu_{68} & \nu_{69} & \nu_{70} & \nu_{71} & \nu_{72} & \nu_{73} & \nu_{74} & \rightarrow \\ 1 & ... & 31 & 30 & 31 & 32 & 34 & 36 & 36 & 36 & 37 & \rightarrow \end{bmatrix}$$

Proposition 3.13 ensures that in the case $c + c' - 2 \leq 2d$, the parameter $m$ equals $c + c' - 2 - g$. Our next concern is to find $m$ in the remaining cases.

From now on we always assume that $c + c' - 2 > 2d$.

**Proposition 3.15** *Assume that $c + c' - 2 > 2d$ and that $d - \ell \in S$. Let $k := 2d - g$ $(s_k = 2d)$. Then the parameter $\nu_{k-1}$ is related to $\nu_k$ as follows:*

$$\nu_{k-1} = \begin{bmatrix} \nu_k - 3 & if & d-1 \notin S & and & d-\ell-1 \in S & (a) \\ \nu_k - 1 & if & d-1 \notin S & and & d-\ell-1 \notin S & (b) \\ \nu_k - 1 & if & d-1 \in S & and & d-\ell-1 \in S & (c) \\ \nu_k + 1 & if & d-1 \in S & and & d-\ell-1 \notin S & (d) \end{bmatrix}$$

Proof. Since $s_{k-1} = 2d - 1$ is odd, clearly $(x, y) \in N_{k-1}$ if and only if either $x > y$, or $y > x$, and $\#N_{k-1} = 2\#\{(x, y) \in N_{k-1} \mid x > y\}$. Let $x > y$, then
either $x \geq c$ so that $(x, y) \in A_c(2d - 1) = \{(x, y) \in S^2 \mid x \geq c,\ \ and\ \ x + y = 2d - 1\}$,
or $x = d$ and $y = d - 1 \in S$. It follows that:

$$\#N_{k-1} = \begin{bmatrix} 2\#A_c(2d-1) & if & d-1 \notin S \\ 2\#A_c(2d-1)+2 & if & d-1 \in S. \end{bmatrix}$$

Now, easily one gets that $(x, y) \in N_k$, with $x \geq y \iff$ either $(x, y) = (d, d)$ or $(x, y) \in A_c(s_k)$.
Hence recalling that $2d - c = d - \ell - 1$ and the fact that $(d, d) \in N_k$, one obtains (see (3.7.7)):

$$\#N_k = \begin{bmatrix} 2\#A_c(2d-1)+1 & if & d-\ell-1 \notin S \\ 2\#A_c(2d-1)+3 & if & d-\ell-1 \in S. \end{bmatrix}$$

The claim follows by combining all the possible cases. $\diamond$

Now we show examples concerning the four cases of the above proposition.

**Example 3.16** (1) Let $S = \{0, 10, 11, 12, 13, 14, 16, 20 \rightarrow\} =< 10, 11, 12, 13, 14, 16 >$.
Then, $c = 20,\ d = c' = 16,\ d' = 14,\ g = 13,\ \ell = 3,$
$d - \ell = 13 \in S,$
$d - 1 = 15 \notin S,\ \ d - \ell - 1 = 12 \in S,$

$d-2=14 \in S, \quad d-\ell-2=11 \in S$
$d-3=13 \in S, \quad d-\ell-3=10 \in S$
$d-4=12 \in S, \quad d-\ell-4= \ 9 \notin S$. Hence $t=4$ and $\{d-1, d-2, d-3\} \cap S \neq \{d-3\}$.
Moreover $c+c'-2=34>2d=32$. In this case we get

$m=2d-g-4=15$, with $s_m=28=2d-4$. In fact the sequence $\{\nu_i\}_{i \in \mathbb{N}}$ is:

$$\begin{bmatrix} 0 & 10 & ... & 27 & 28=s_m & 29 & 30 & 31 & 32=2d & 33 & \rightarrow \\ \nu_0 & \nu_1 & ... & \nu_{14} & \nu_{15} & \nu_{16} & \nu_{17} & \nu_{18} & \nu_{19=2d-g} & \nu_{20} & \rightarrow \\ 1 & 2 & ... & 6 & 5 & 4 & 6 & 6 & 9 & 10 & \rightarrow \end{bmatrix}$$

(2) Let $S=\{0, 10, 11, 13, 14, 16, 20 \rightarrow\}$, generated by $< 10, 11, 13, 14, 16 >$.
So: $c=20, \ d=c'=16, \ d'=14, \ g=14, \ \ell=3, \ d-\ell \in S, \ d-1 \notin S, \ d-\ell-1 \notin S$.
$d-1=15 \notin S, \quad d-\ell-1 = \notin S,$
$d-2=14 \in S, \quad d-\ell-2=11 \in S$
$d-3=13 \in S, \quad d-\ell-3=10 \in S$
$d-4=12 \notin S, \ d-\ell-4 \notin S$
$d-5=11 \in S, \ d-\ell-5 \notin S$. Hence $t=5$ and $\{d-1, d-2, d-3\} \cap S \neq \{d-2\}$. Moreover $c+c'-2=34>2d=32$. In this case we get

$m=2d-g-5=13$, with $s_m=27$. In fact the sequence $\{\nu_i\}_{i \in \mathbb{N}}$ is:

$$\begin{bmatrix} 0 & 10 & ... & 27=s_m & 28 & 29 & 30 & 31 & 32=2d & 33 & \rightarrow \\ \nu_0 & \nu_1 & ... & \nu_{13} & \nu_{14} & \nu_{15} & \nu_{16} & \nu_{17} & \nu_{18=2d-g} & \nu_{20} & \rightarrow \\ 1 & 2 & ... & \mathbf{6} & 3 & 4 & 6 & 6 & 7 & 8 & \rightarrow \end{bmatrix}$$

(3) Let $S=\{0, 10, 11, 12, 13, 15, 16, 20 \rightarrow\}$, generated by $< 10, 11, 12, 13, 15, 16 >$. So: $c=20, d=16$, $c'=15, \ d'=13, \ g=13, \ \ell=3, \ d-\ell \in S, \ d-1 \in S, \ d-\ell-1 \in S, \quad t=0$.
Moreover $c+c'-2=33>2d=32$. In this case we get

$m=2d-g-4=15$, with $s_m=28$. In fact the sequence $\{\nu_i\}_{i \in \mathbb{N}}$ is

$$\begin{bmatrix} 0 & 10 & ... & 27 & 28=s_m & 29 & 30 & 31 & 32=2d & 33 & \rightarrow \\ \nu_0 & \nu_1 & ... & \nu_{14} & \nu_{15} & \nu_{16} & \nu_{17} & \nu_{18} & \nu_{19=2d-g} & \nu_{20} & \rightarrow \\ 1 & 2 & ... & 6 & \mathbf{6} & 4 & 5 & 8 & 9 & 10 & \rightarrow \end{bmatrix}$$

(4) Let $S=\{0, 10, 11, 13, 15, 16, 20 \rightarrow\} =< 10, 11, 12, 13, 15, 16 >$. Then,
$c=20, \ d=16, \ c'=15, \ d'=13, \ g=14, \ \ell=3, \ d-\ell \in S, \ d-1 \in S, \ d-\ell-1 \notin S$. Hence $t=1$. Moreover $c+c'-2=33>2d=32$. In fact the sequence $\{\nu_i\}_{i \in \mathbb{N}}$ is

$$\begin{bmatrix} 0 & 10 & ... & 30 & 31=s_m & 32=2d & 33 & 34 & \rightarrow \\ \nu_0 & \nu_1 & ... & \nu_{16} & \nu_{17} & \nu_{18=2d-g} & \nu_{19} & \nu_{20} & \rightarrow \\ 1 & 2 & ... & 5 & \mathbf{8} & 7 & 8 & 8 & \rightarrow \end{bmatrix}$$

Hence $m=2d-g-1=17$, with $s_m=31$.

By combining Proposition 3.15 and Proposition 3.9 we obtain

**Corollary 3.17** *Suppose $c+c'-2>2d$. Then,*

(1) $m=2d-g-1 \iff t=1$.

(2) $m<2d-g-1 \iff t>1$.

With similar techniques we can go further in the study of the remaining cases.

**Proposition 3.18** *Assume that $c+c'-2>2d$ and that $d-\ell \in S$. Let $k:=2d-g-1$ $(s_k=2d-1)$ $u:=d-\ell-2$, $A:=A_c(2d-2)=A_c(s_{k-1})$ and $A':=\{(x+1, y) \mid (x, y) \in A\}=A_{c+1}(2d-1)$.*

*Then $\nu_{k-1} = \nu(2d-2)$ is related to $\nu_k = \nu(2d-1)$ as the following scheme shows.*

| $d-1$ | $d-2$ | $u$ | $\{(x,y) \in N_{k-1} \mid x \geq y\}$ | $\{(x,y) \text{ in } N_k \mid x \geq y\}$ | $\nu_{k-1}$ |
|---|---|---|---|---|---|
| $\times$ | 0 | $\times$ | $\{(d-1,d-1)\} \cup A$ | $\{(d,d-1),(c,u)\} \cup A'$ | $\nu_k - 3$ |
| 0 | 0 | $\times$ | $A$ | $\{(c,u)\} \cup A'$ | $\nu_k - 2$ |
| $\times$ | 0 | 0 | $\{(d-1,d-1)\} \cup A$ | $\{(d,d-1)\} \cup A'$ | $\nu_k - 1$ |
| $\times$ | $\times$ | $\times$ | $\{(d,d-2),(d-1,d-1)\} \cup A$ | $\{(d,d-1),(c,u)\} \cup A'$ | $\nu_k - 1$ |
| 0 | 0 | 0 | $A$ | $A'$ | $\nu_k$ |
| 0 | $\times$ | $\times$ | $\{(d,d-2)\} \cup A$ | $\{(c,u)\} \cup A'$ | $\nu_k$ |
| $\times$ | $\times$ | 0 | $\{(d,d-2),(d-1,d-1)\} \cup A$ | $\{(d,d-1)\} \cup A'$ | $\nu_k + 1$ |
| 0 | $\times$ | 0 | $\{(d,d-2)\} \cup A$ | $A'$ | $\nu_k + 2$ |

*Here for an integer $s$ we write respectively "$\times$" if $s \in S$ or "0" if $s \notin S$.*

Proof. To find the sets $N_k$ and $N_{k-1}$, one notes that $(x,y) \in N_k$ (respectively $N_{k-1}$), with $x \geq y$, implies that either $x \in \{d-2, d-1, d\}$, or $(x,y) \in A_c(s_k)$ (resp. $A_c(s_{k-1})$). Since $A_c(2d-1) = A'$ if $u \notin S$, and $A_c(2d-1) = A' \cup \{(c,u)\}$, if $u \in S$ by (3.7), one gets the above scheme. $\diamond$

From (3.18), (3.9) and (3.17) we deduce the following corollary.

**Corollary 3.19** *Suppose $c + c' - 2 > 2d$. Then,*

(1) $m = 2d - g - 2 \iff t = 2.$

(2) $m < 2d - g - 2 \iff t > 2.$

We can apply the same arguments once more, but we'll see that new cases arise.

**Proposition 3.20** *Assume that $c + c' - 2 > 2d$ and that $d - \ell \in S$. Let $k := 2d - g - 2$ $(s_k = 2d-2)$, $u := d - \ell - 3$, $A := A_c(2d-3) = A_c(s_{k-1})$, $A' := \{(x+1,y) \mid (x,y) \in A\} = A_{c+1}(2d-2)$, $B(2d-3) := \{(x,y) \in N_{k-1} \mid x \geq y\} \setminus A$, $B'(2d-2) := \{(x,y) \in N_k \mid x \geq y\} \setminus A'$. Then the parameters $\nu_{k-1} = \nu(2d-3)$ and $\nu_k = \nu(2d-2)$ are related as follows:*

| $d-1$ | $d-2$ | $d-3$ | $u$ | $B(2d-3)$ | $B'(2d-2)$ | $\nu_{k-1}$ |
|---|---|---|---|---|---|---|
| 0 | $\times$ | 0 | $\times$ | | $\{(d,d-2),(c,u)\}$ | $\nu_k - 4$ |
| $\times$ | 0 | 0 | $\times$ | | $\{(d-1,d-1),(c,u)\}$ | $\nu_k - 3$ |
| $\times$ | $\times$ | 0 | $\times$ | $\{(d-1,d-2)\}$ | $\{(d,d-2),(d-1,d-1),(c,u)\}$ | $\nu_k - 3$ |
| 0 | $\times$ | 0 | 0 | | $\{(d,d-2)\}$ | $\nu_k - 2$ |
| 0 | 0 | 0 | $\times$ | | $\{(c,u)\}$ | $\nu_k - 2$ |
| 0 | $\times$ | $\times$ | $\times$ | $\{(d,d-3)\}$ | $\{(d,d-2),(c,u)\}$ | $\nu_k - 2$ |
| $\times$ | 0 | $\times$ | $\times$ | $\{(d,d-3)\}$ | $\{(d-1,d-1),(c,u)\}$ | $\nu_k - 1$ |
| $\times$ | $\times$ | $\times$ | $\times$ | $\{(d,d-3),(d-1,d-2)\}$ | $\{(d,d-2),(d-1,d-1),(c,u)\}$ | $\nu_k - 1$ |
| $\times$ | 0 | 0 | 0 | | $\{(d-1,d-1)\}$ | $\nu_k - 1$ |
| $\times$ | $\times$ | 0 | 0 | $\{(d-1,d-2)\}$ | $\{(d,d-2),(d-1,d-1)\}$ | $\nu_k - 1$ |
| 0 | 0 | 0 | 0 | | | $\nu_k$ |
| 0 | 0 | $\times$ | $\times$ | $\{(d,d-3)\}$ | $\{(c,u)\}$ | $\nu_k$ |
| 0 | $\times$ | $\times$ | 0 | $\{(d,d-3)\}$ | $\{(d,d-2)\}$ | $\nu_k$ $(a)$ |
| $\times$ | $\times$ | $\times$ | 0 | $\{(d,d-3),(d-1,d-2)\}$ | $\{(d,d-2),(d-1,d-1)\}$ | $\nu_k + 1$ |
| $\times$ | 0 | $\times$ | 0 | $\{(d,d-3)\}$ | $\{(d-1,d-1)\}$ | $\nu_k + 1$ |
| 0 | 0 | $\times$ | 0 | $\{(d,d-3)\}$ | | $\nu_k + 2$ |

*Here for an integer $s$ we write respectively "$\times$" if $s \in S$, "0" if $s \notin S$.*

From (3.20), (3.9), (3.17 and(3.19.2)) we deduce the following corollary.

**Corollary 3.21** *Suppose $c + c' - 2 > 2d$. Then,*

(1) $m = 2d - g - 3 \iff t = 3$ *and* $\{d-1, d-2\} \cap S \neq \{d-2\}$.

(2) $m < 2d - g - 3$ *if* $t > 3$, *or* $t = 3$ *and* $\{d-1, d-2\} \cap S = \{d-2\}$.

To study the case $t = 4$, by using the same tools as in the previous cases we obtain a new scheme with $s_k = 2d - 3$ and $s_{k-1} = 2d - 4$. We omit some detail, but the result is the following:

$$
\left[\begin{array}{ccccc|ll}
d-1 & d-2 & d-3 & d-4 & d-\ell-4 & \nu_{k-1} & \\
 & & & & & & \\
0 & 0 & \times & 0 & \times & \nu_k - 4 & \\
0 & \times & \times & 0 & \times & \nu_k - 3 & \\
\times & \times & 0 & 0 & \times & \nu_k - 3 & \\
\times & \times & \times & 0 & \times & \nu_k - 3 & \\
0 & 0 & \times & 0 & 0 & \nu_k - 2 & \\
\times & 0 & 0 & 0 & \times & \nu_k - 2 & \\
0 & 0 & 0 & 0 & \times & \nu_k - 2 & \\
0 & 0 & \times & \times & \times & \nu_k - 2 & \\
\times & 0 & \times & 0 & \times & \nu_k - 2 & \\
0 & \times & \times & 0 & 0 & \nu_k - 1 & \\
0 & \times & \times & \times & \times & \nu_k - 1 & \\
\times & \times & 0 & \times & \times & \nu_k - 1 & \\
\times & \times & \times & \times & \times & \nu_k - 1 & \\
0 & \times & 0 & 0 & \times & \nu_k - 1 & \\
\times & \times & 0 & 0 & 0 & \nu_k - 1 & \\
\times & \times & \times & 0 & 0 & \nu_k - 1 & \\
\times & 0 & 0 & 0 & 0 & \nu_k & \\
0 & 0 & 0 & 0 & 0 & \nu_k & \\
\times & 0 & 0 & \times & \times & \nu_k & \\
\times & 0 & \times & \times & \times & \nu_k & \\
0 & 0 & 0 & \times & \times & \nu_k & \\
\times & 0 & \times & 0 & 0 & \nu_k & \\
0 & \times & 0 & \times & \times & \nu_k + 1 & (b) \\
0 & \times & 0 & 0 & 0 & \nu_k + 1 & (c) \\
0 & 0 & \times & \times & 0 & \nu_k & (a) \\
0 & \times & \times & \times & 0 & \nu_k + 1 & \\
\times & \times & 0 & \times & 0 & \nu_k + 1 & \\
\times & \times & \times & \times & 0 & \nu_k + 1 & \\
\times & 0 & 0 & \times & 0 & \nu_k + 2 & \\
\times & 0 & \times & \times & 0 & \nu_k + 2 & \\
0 & 0 & 0 & \times & 0 & \nu_k + 2 & \\
0 & \times & 0 & \times & 0 & \nu_k + 3 &
\end{array}\right] \tag{4}
$$

From this last scheme we deduce the following corollary.

**Corollary 3.22** (1) *If* $t = 4$, *then* $m \leq 2d - g - 4$ *and*
$m = 2d - g - 4$ *if and only if* $\{d-1, d-2, d-3\} \cap S \neq \{d-3\}$.

(2) *If* $t > 4$, *then* $m \leq 2d - g - 4$ *and*

$m = 2d - g - 4$ *if and only if*

$$\{d-1, d-2, d-3\} \cap S = \{d-2\} \text{ and } \left[\begin{array}{cl} \textit{either} & d-4 \in S,\ d-\ell-4 \in S, \\ \textit{or} & d-4 \notin S,\ d-\ell-4 \notin S. \end{array}\right.$$

By summarizing all the previous results we obtain Theorem 3.1 and Theorem 3.2.

Cases $(a),(b),(c)$ in the above table (4) show that when $t$ is greater than or equal to 4 we cannot always predict the value of $m$: see also the examples below.

**Example 3.23** To evaluate $m$ in the following examples we need (3.9.1)
(1) Conditions $(t=4,\ m=2d-g-4)$ of (3.22.1) are satisfied in Example 3.16.1.
(2) The conditions of case $(a)$ in table (4) and $(t=4,\ m<2d-g-4)$ of (3.22.1) are satisfied for instance by $S=\{0,10,12,13,16,20\rightarrow\}=<10,12,13,16,21,27>$.
We have: $c=20,\ d=c'=16,\ d'=13,\ g=15,\ \ell=3,$
$d-\ell=13\in S,$
$d-1=15\notin S,\quad d-\ell-1=12\in S,$
$d-2=14\notin S,\quad d-\ell-2=11\notin S$
$d-3=13\in S,\quad d-\ell-3=10\in S$
$d-4=12\in S,\quad d-\ell-4=\ 9\notin S.$ Hence $t=4$ and $\{d-1,d-2,d-3\}\cap S=\{d-3\}$.
Moreover $c+c'-2=34>2d=32$.
In this case we get $m=2d-g-6=11$, with $s_m=26<2d-4$. In fact the sequence $\{\nu_i\}_{i\in\mathbb{N}}$ is:

$$\begin{bmatrix} 0 & 10 & ... & 26=s_m & 27 & 28 & 29 & 30 & 31 & 32=2d & 33 & \rightarrow \\ \nu_0 & \nu_1 & ... & \nu_{11}=\nu_m & \nu_{12} & \nu_{13} & \nu_{14} & \nu_{15} & \nu_{16} & \nu_{17} & \nu_{18} & \rightarrow \\ 1 & 2 & ... & \mathbf{5} & 2 & 4 & 4 & 4 & 4 & 7 & 8 & \rightarrow \end{bmatrix}$$

(3) The conditions of case $(b)$ in table (4) and $(t>4,\ m=2d-g-4)$ of (3.22.2) are satisfied for instance by $S=\{0,10,11,13,15,17,20\rightarrow\}=<10,11,13,15,17,29>$.
$c=20,\ d=c'=17,\ d'=15,\ g=14,\ \ell=2,$
$d-1=16\notin S,\qquad d-\ell-1=14\notin S,$
$d-2=d-\ell=15\in S,\quad d-\ell-2=13\in S$
$d-3=14\notin S,$
$d-4=13\in S,\qquad d-\ell-4=\ 11\in S$
$d-5=12\notin S$
$d-6=11\in S,\qquad d-\ell-6=9\notin S.$ Hence $t=6$ and $\{d-1,d-2,d-3\}\cap S=\{d-2\}$.
Moreover $c+c'-2=35>2d=34$.
In this case we get $m=2d-g-4=16$, with $s_m=30=2d-4$. In fact the sequence $\{\nu_i\}_{i\in\mathbb{N}}$ is:

$$\begin{bmatrix} 0 & 10 & ... & 30=s_m & 31 & 32 & 33 & 34=2d & 35 & \rightarrow \\ \nu_0 & \nu_1 & ... & \nu_{16}=\nu_m & \nu_{17} & \nu_{18} & \nu_{18} & \nu_{20} & \nu_{21} & \rightarrow \\ 1 & 2 & ... & \mathbf{7} & 6 & 8 & 8 & 9 & 10 & \rightarrow \end{bmatrix}$$

(4) The conditions of case $(c)$ in table (4) and $(t>4,\ m=2d-g-4)$ of (3.22.2) are satisfied for instance by $S=\{0,13,15,18,20,26\rightarrow\}=<13,15,18,20,26,27,29,32,34,37>$.
$c=26,\ d=c'=20,\ d'=18,\ g=21,\ \ell=5,$
$d-1=19\notin S,\qquad d-\ell-1=14\notin S,$
$d-2=18\in S,\qquad d-\ell-2=13\in S,$
$d-3=17\notin S,$
$d-4=16\notin S,\qquad d-\ell-4=\ 11\notin S,$
$d-5=d-\ell=15\in S,$
$d-6=14\notin S,$
$d-7=13\in S,\qquad d-\ell-7=8\notin S.$ Hence $t=7$ and $\{d-1,d-2,d-3\}\cap S=\{d-2\}$.
Moreover $c+c'-2=35>2d=34$.
In this case we get $m=2d-g-4=15$, with $s_m=36=2d-4$. In fact the sequence $\{\nu_i\}_{i\in\mathbb{N}}$ is:

$$\begin{bmatrix} 0 & 13 & ... & 35 & 36=s_m & 37 & 38 & 39 & 40=2d & 41 & \rightarrow \\ \nu_0 & \nu_1 & ... & \nu_{14} & \nu_{15}=\nu_m & \nu_{16} & \nu_{17} & \nu_{18} & \nu_{19} & \nu_{20} & \rightarrow \\ 1 & 2 & ... & 4 & \mathbf{3} & 2 & 4 & 4 & 5 & 6 & \rightarrow \end{bmatrix}$$

(5) Let $\ell>1$ and let $S_\ell=\{0,2\ell+1,3\ell+1,4\ell+2,\rightarrow\}$.
We have $c=4\ell+2,\ c'=d=3\ell+1,\ d'=2\ell+1,\ g=4\ell-1$. Clearly $t=\ell,\ \ c+c'-2=7\ell+1>2d=6\ell+2$ and the sequence $\{\nu_i\}_{i\in\mathbb{N}}$ is:

$$\begin{bmatrix} 0 & 2\ell+1 & ... & 5\ell+2=s_{2d-t} & 5\ell+3 & ... & 6\ell+1 & 2d & 6\ell+3 & \rightarrow \\ \nu_0 & \nu_1 & ... & \nu_{2d-t-g} & \nu_{m+1} & ... & \nu_{2d-1} & \nu_{2d} & \nu_{2d+1} & \rightarrow \\ 1 & 2 & ... & \mathbf{4} & 2 & 2 & 2 & 3 & 4 & \rightarrow \end{bmatrix}$$

Hence we get $m = 2d - g - t$.

**Remark 3.24** (1) Examples (3) and (4) in (3.23) show that the formula found for $t \leq 4$ : $m = 2d - g - i$ if and only if $t = i$ and $\{d-1, ..., d-i+1\} \cap S \neq \{d-i+1\}$ doesn't hold in general for $t > 4$. However this formula is true for every $t > 1$ in the family $S_\ell$ of example (3.23.5).

(2) Examples (4) and (5) in (3.23) show that the inequality $m \leq 2d - g - 4$ when $t > 4$ in Corollary 3.22.2 can be strict or not.

# 4 Classes of examples.

In this section we show some classes of acute semigroups, that are new with respect to the ones studied in [1]. For such semigroups we find also the value of the parameter $m$ as in (2.3).

## 4.1 Semigroups generated by arithmetic sequences.

Semigroups generated by arithmetic or almost arithmetic sequences often arise among the Weierstrass semigroups (see next Example 4.3). We shall prove that all semigroups generated by an arithmetic sequence are acute. First we recall some definitions and fix some new notations.

**Definition 4.1** We say that the semigroup $S$ as in 2.1 is *generated by an arithmetic sequence* ($AS$ for brevity) if

$S =< m_0, m_1, ..., m_{p+1} >$, where $m_0 \geq 2$, $m_i = m_0 + \rho\, i$, $\forall\, i = 1, ..., p+1$, and $GCD(\rho, m_0) = 1$.

When $\rho = 1$, we say that $S$ *is generated by an interval* (see e.g. [5]).

We shall denote by $q$, $r$ the integers such that $m_0 - 2 = q(p+1) + r$, $(0 \leq r \leq p)$.

**Definition 4.2** We say that a semigroup $S$ is *generated by an almost arithmetic sequence* ($AAS$ for brevity) if

$S =< m_0, m_1, ..., m_{p+1}\ , n >$ *with* $m_0 \geq 2$, $m_i = m_0 + \rho\, i$, $\forall\, i = 1, ..., p+1$, *and* $GCD(\rho, m_0, n) = 1$.

Some semigroups arising from AG codes are $AS$ or $AAS$, as shown in the following example.

**Example 4.3** (1) $S_1 =< 3, 5, 7 >$ ($AS$) is the Weierstrass semigroup at $P_0 = (0 : 0 : 1)$ of the *Klein quartic* $\subseteq I\!P^2$ defined by the equation $X_0^3 X_1 + X_1^3 X_2 + X_0 X_2^3 = 0$, over the field $I\!F$ having cardinality $q$ with $gcd(q, 7) = 1$ [6, Example 2.14].
(2) $S_2 =< 4, 5 >$ ($AS$), $S_3 =< 4, 7, 10, 13 >$ ($AS$), $S_\ell = \{0, 6 \rightarrow\} \setminus \{\ell\}$, for each $6 \leq \ell \leq 11$ ($AS$ if and only if $\ell = 6$ or $\ell = 11$, $AAS$ if and only if $\ell = 10$),
are the possible Weierstrass semigroups for a *plane non singular projective quintic* [8, Section 3].
(3) If $S$ has $g$ gaps, it is easy to see that:
If $2 \in S$, then $S =< 2, 2g+1 >$ ($AS$) (*hyperelliptic semigroup*, see e.g. [2, Example 3 ]).
If $2 \notin S$ and $2 \leq g < 4$, then $S$ is $AS$ or $AAS$.
If $2 \notin S$ and $g = 4$ then $S$ is $AS$ or $AAS$, if and only if $S \neq \{0, 4, 6 \rightarrow\} (=< 4, 6, 7, 9 >)$.
In fact all the possible semigroups are:
When $g = 2$: $< 3, 4, 5 >$;
When $g = 3$: $< 3, 4 >$, $< 3, 5, 7 >$, $< 4, 5, 6, 7 >$;
When $g = 4$: $< 3, 5 >$, $< 3, 7, 8 >$ (AAS), $< 4, 6, 7, 9 >$, $< 5, 6, 7, 8, 9 >$, $< 4, 6, 7 >$ (AAS).

We recall from [10] the following facts which hold for $AS$ semigroups.

**Proposition 4.4** [10, Prop. 2.5.2-3 and Lemma 2.6]. *Assume $S$ is generated by an arithmetic sequence as in (4.1) and let $H_2$ be the set of gaps of the second type as in (2.1). Then,*

(1) *The conductor can be written as* $c = (m_0 - 1)(\rho + q) + q + 1 = m_{r+1} + qm_{p+1} - m_0 + 1.$

(2) $S$ *is symmetric if and only if* $r = 0.$

(3) *If* $r \geq 1$*, the gaps of the second type are* $\{m_i + qm_{p+1} - m_0 - jm_{p+1}, \ \ 1 \leq i \leq r, \ \ 0 \leq j \leq q\} = \{m_i + qm_{p+1} - m_0, ..., m_i + (q-j)m_{p+1} - m_0, ..., m_i + m_{p+1} - m_0, \ \ 1 \leq i \leq r, \ \ 0 \leq j \leq q\}.$

*and the greatest gap in* $H_2$ *is* $\mu = m_r + qm_{p+1} - m_0 = c - \rho - 1.$

**Lemma 4.5** *Let* $S$ *be an AS semigroup. Suppose* $r \geq 1$ *and let* $\mu$ *be as in (4.4.3) above. Then,* $\mu = c - 2 \iff \rho = 1$*, i.e.* $S$ *is generated by an interval.*

Proof. It follows by (4.4.3). $\diamond$

**Lemma 4.6** *Let* $S$ *be an AS semigroup. Assume* $\rho = 1$*. We have:*

$$c = (q+1)m_0, \quad d = qm_{p+1} = c - r - 2, \quad c' = qm_0 = c - m_0, \quad d' = (q-1)m_{p+1}.$$

(The equality $c = (q+1)m_0$, is proved in [5, Corollary 5])

Proof. If $\rho = 1$, since $m_0 - 2 = q(p+1) + r$ (4.1), using (4.4.1) we get: $c = m_{r+1} + qm_{p+1} - m_0 + 1 = m_0 + r + 1 + q(m_0 + p + 1) - m_0 + 1 = q(p+1) + r + qm_0 + 2 = m_0 - 2 + qm_0 + 2 = (q+1)m_0.$ Moreover, since $S = \bigcup_{k \geq 0}\{km_0, km_0 + 1, km_{p+1}\}$ (see e.g [1, Lemma 4.2]), one has $d = qm_{p+1}$, so $c - d = qm_0 + m_0 - qm_0 - q(p+1) = r + 2$, and we can easily prove the other statements. $\diamond$

**Lemma 4.7** *Let* $S$ *be an AS semigroup and let* $H_2$ *the set of gaps of the second kind of* $S$*. Assume* $\rho \geq 2$*.*

(1) *If* $\sigma \in \mathbb{N}$*,* $1 \leq \sigma \leq \rho$*, then* $c - \sigma \notin H_2$*.*

(2) $d = c - 2.$

(3) $c' = \left[\begin{array}{lll} c - m_0 & if & r = 0 \\ c - m_0 & if & r \geq 1 \text{ and } \rho > m_0 \\ c - \rho & if & r \geq 1 \text{ and } \rho < m_0. \end{array}\right.$

Proof. (1) If $r = 0$, then $H_2 = \emptyset$ by (4.4.2) and (2.1). When $r \geq 1$, let $\mu$ be as in (4.4.3), and let $1 \leq \sigma \leq \rho$. Then $\mu = c - (\rho + 1) < c - \sigma$ and it is enough to recall that $\mu$ is the greatest gap in $H_2$.
(2). It follows by (4.5) and (3.6.5).
(3). If $r = 0$, then $R$ is a symmetric semigroup (4.4.2), hence $c' = c - m_0$, by (3.6.2).
Assume $r \geq 1$. If $\rho < m_0$, the interval $[c - \rho, c - 2]$ is contained in $S$. Indeed, for $s \in [c - \rho, c - 2]$, one has $s = c - \alpha$ with $2 \leq \alpha \leq \rho < m_0$, and so $c - \alpha \notin H$, by (3.6.3) and by (1). Since $c - \rho - 1 = \mu \notin S$ (4.4.3), we obtain that $c' = c - \rho$.
If $\rho > m_0$, then $[c - m_0, c - 2] \subseteq S$. In fact, if $s \in [c - m_0, c - 2]$, one has : $s = c - \beta$, with $2 \leq \beta \leq m_0 < \rho$, and so $s \notin H$ by (3.6.3) and by (1). Since $c - m_0 - 1 \notin S$ (it is in $H_1$), we get $c' = c - m_0$. $\diamond$

**Theorem 4.8** *Let* $S \subseteq \mathbb{N}$ *be a semigroup generated by an arithmetic sequence.*

(1) $S$ *is an acute semigroup. In particular, if* $\rho \geq 2$*, then* $d = c - 2$*.*

(2) *Let* $m$ *be as in (2.3). Then:* $\left[\begin{array}{lll} m = c + c' - 2 - g & if & \rho \geq 2 \\ m = c + c' - 2 - g & if & \rho = 1 \text{ and } 0 \leq r \leq \left[\frac{m_0 - 2}{2}\right] \\ m = 2d - g & & \text{otherwise.} \end{array}\right.$

Proof. (1) S is acute by (4.4.2) and (3.6.2), if $r = 0$; by [1, Prop. 5.9.4], if $r \geq 1$ and $\rho = 1$ ; by (4.7.2) and (3.6.1), if $r \geq 1$ and $\rho \geq 2$.
(2). If ($\rho = 1$ and $r = 0$), or $\rho \geq 2$, then $d = c-2$ by (4.6) and (4.7.2); and so $c+c'-2 \leq c+d-2 = 2d$.
If $\rho = 1$ and $r \geq 1$, then $c' = c - m_0$, $d = c - r - 2$ (4.6); so $c + c' - 2 = 2c - m_0 - 2$, and $c + c' - 2 \leq 2d = 2c - 2r - 4 \Longleftrightarrow m_0 \geq 2r + 2 \Longleftrightarrow r \leq \left[\frac{m_0-2}{2}\right]$. $\diamond$

We remark that there are semigroups generated by an almost arithmetic sequence,which are not acute, as shown in the following example.

**Example 4.9** Let $S = \{0, 9, 16, 17, 18, 23, 25, 26, 27, 30, 32, 33, 34, 35, 36, 39 \rightarrow\} =< 9, 16, 17, 23, 30 >$. Then $c = 39$, $d = 36$, $c' = 32$, $d' = 30$. Hence: $c - d = 3 > c' - d' = 2$.

In the next subsection we shall prove that every semigroup generated by 3 elements (hence generated by an almost arithmetic sequence) is acute.

## 4.2 Semigroups of Cohen Macaulay type 2 or 3.

We conclude the section by proving that a semigroup $S$ as in (2.1) with $\tau = 2$, where $\tau$ is the CM-type (see 2.1), is acute. In particular all the semigroups generated by three elements are acute. We give also partial answers if $\tau = 3$. We start with some preliminary lemmas.

**Remark 4.10** (1) $d \geq c' \geq c - e$.

(2) *Every gap* $\sigma \geq c - e$ *belongs to* $(S(1) \setminus S)$.

(3) $\{d + 1, ..., d + \ell\} = \{c - \ell, ..., c - 1\} \subseteq S(1) \setminus S$, *and* $\ell \leq \tau$.

(4) $\{c - e - \ell, ..., c - e - 1\} \subseteq \mathbb{N} \setminus S$.

Proof. (1). Observe that $c - 1 \notin S \Longrightarrow c - 1 - e \notin S$, then(1) is clear since $c' - 1 \notin S$ and $[c', d] \subseteq S$.
(2). For any gap $\sigma \geq c - e$, we immediately get that $\sigma + s \geq c$ for all $s \in S \setminus \{0\}$, hence $\sigma \in S(1) \setminus S$.
(3). It follows from (2), since $d \geq c - e$, by (1).
(4). Immediate, since $c - e - \ell + i + e \notin S$, for every $i$, $0 \leq i \leq \ell - 1$.

**Lemma 4.11** *The following hold:*

(1) $d \geq c - 1 - \tau$.

(2) *If* $d = c - 1 - \tau$ *then* $c' = c - e$.

(3) *If either* $c' = c - e$ *or* $c' = c - e + 1$, *then* $S$ *is acute.*

Proof. (1). Since $\ell \leq \tau$ by (4.10.3), we obtain $d = c - \ell - 1 \geq c - \tau - 1$, as desired.
(2). Note that $d = c - 1 - \tau \Longrightarrow [d + 1, ..., d + \tau] = S(1) \setminus S$, by (4.10.3); on the other hand we cannot have other gaps greater than $c - e$ by (4.10.2), hence $c' \leq c - e$ and the result follows by (4.10.1).
(3). When $c' = c - e$, from (4.10.4) we get $d' \leq c - e - \ell - 1$. Hence $c' - d' \geq \ell + 1 = c - d$ and this means that $S$ is acute (see 2.1).
When $c' = c - e + 1$, by definition $c - e$ is a gap of $S$; moreover $\{c - e - \ell, ..., c - e - 1\}$ are gaps by (4.10.4), hence $d' \leq c - e - \ell - 1$ and we conclude the proof. $\diamond$.

**Remark 4.12** There are acute semigroups with $c' \notin \{c - e, c - e + 1\}$. For example, let $S =< 7, 11, 15 >$. Then $S$ is acute with $c' = 35$, $c - e = 32$.

**Proposition 4.13** *Let* $S$ *be a non-ordinary semigroup.*

(1) *If* $\tau = 2$, *then* $S$ *is acute and either* $d = c - 2$, *or* $d = c - 3$.

*Further, if* $d = c - 3$ *and* $e = 3$, *then* $c + c' - 2 = 2d + 1$, *otherwise* $c + c' - 2 \leq 2d$.

(2) *If* $S$ *is generated by* 3 *elements, then* $\tau \leq 2$ *and* $S$ *is acute.*

Proof. (1) From (4.11.1) we deduce that either $d = c - 2$, or $d = c - 3$. In the first case $S$ is obviously acute (3.6.1) and $c + c' - 2 \leq c + d - 2 = 2d$.
Let $d = c - 3$. First note that $e \geq 3$, otherwise $d = c - 2$. Then deduce that $S$ is acute by (4.11.2) and (4.11.3). When $e = 3$, one has $c' = d = c - 3$, so $c + c' - 2 = 2c - 5 = 2d + 1$.
If $e \geq 4$ one has $c' \leq d - 1$ by (4.11.2), so $c + c' - 2 \leq c + d - 3 = 2d$.
(2) When $S$ is minimally generated by 3 elements, then $S$ is generated by an almost arithmetic sequence and $\tau \leq 2$, ( see [11, Props. 3.3, 4.6, 5.6]). So the result follows by (1) if $\tau = 2$ and by (3.6.2) if $\tau = 1$ (since $\tau = 1$ means that $S$ is symmetric). $\diamond$

**Corollary 4.14** *Let $\tau = 2$. Then,*
$m = 2d - g$ *if* $d = c - 3$ *and* $e = 3$, $\qquad m = c + c' - 2 - g$ *otherwise.*

Proof. Since $S$ is acute, by (4.13.1), applying (2.5) we get $m = min\{c + c' - 2 - g,\ 2d - g\}$. $\diamond$

**Remark 4.15** If $\tau = 3$, in general $S$ is not acute. For example: $S =< 7, 10, 13, 15 >$. Here $\tau = 3,\ c = 20,\ d = 17,\ c' = d,\ d' = 15$. In case $\tau = 3$, we can however prove the following facts.

**Proposition 4.16** *Let $S$ be as in (4.13) and let $\tau = 3$. Then,*

(1) $c - 4 \leq d \leq c - 2$.

(2) *If* $d = c - 2$, *then* $S$ *is acute and* $m = c + c' - 2 - g$.

(3) *If* $d = c - 4$, *then* $S$ *is acute and* $m = \left[ \begin{array}{cc} c + c' - 2 - g & \iff e \geq 6 \\ 2d - g & \iff e \leq 5. \end{array} \right.$

(4) *If* $d = c - 3$, *then* $S$ *is acute if and only if* $d' \leq c' - 3$.

Proof. (1) follows directly from (4.11.1).
(2) If $d = c - 2$ see (3.6.1).
(3) If $d = c - 4$ then $d = c - 1 - \tau$ and so $S$ is acute and $c' = c - e$ by (4.11). Moreover $c + c' - 2 = 2c - e - 2 = 2d - e + 6$, and we are done.
(4) When $d = c - 3$ the required inequality holds if and only if $S$ is acute, by definition. $\diamond$

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