On the order bound of one-point algebraic geometry codes.

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[1] **Abstract.** Let $S = \{s_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ be a numerical semigroup. For each $i \in \mathbb{N}$, let $\nu(s_i)$ denote the number of pairs $(s_i - s_j, s_j) \in S^2$: it is well-known that there exists an integer m such that the sequence $\{\nu(s_i)\}_{i\in\mathbb{N}}$ is non-decreasing for $i > m$. The problem of finding m is solved only in special cases. By way of a suitable parameter t, we improve the known bounds for m and in several cases we determine m explicitely. In particular we give the value of m when the Cohen-Macaulay type of the semigroup is three or when the multiplicity is lower or equal to six. When S is the Weierstrass semigroup of a family $\{\mathcal{C}_i\}_{i\in\mathbb{N}}$ of one-point algebraic geometry codes, these results give better estimates for the order bound on the minimum distance of the codes $\{\mathcal{C}_i\}$.

Index Terms. Numerical semigroup, Weierstrass semigroup, algebraic geometry code, order bound on the minimum distance.

1 Introduction

Let $S \subseteq \mathbb{N}$ be a numerical semigroup, $S = \{s_i\}_{i\in\mathbb{N}}$ and let c, c', d, d' denote respectively the conductor, the subconductor, the dominant of the semigroup and the greatest element in S preceding c' (when $d > 0$), as in Setting 2.1. Further let ℓ be the number of gaps of S greater than d and g the genus of S. For $s_i \in S$, call $\nu(s_i)$ the number of pairs $(s_i - s_j, s_j) \in S^2$: when S is the Weierstrass semigroup of a family $\{\mathcal{C}_i\}_{i\in\mathbb{N}}$ of one-point algebraic geometry (AG) codes (see, e.g. [3]), Feng and Rao proved that the minimum distance of the code $\mathcal{C}_i$ can be bounded by the so called order bound, $d_{ORD}(C_i) := min\{\nu(s_j) : \ j \geq i+1\}$ (see [2]). It is well-known that the sequence $\{\nu(s_i)\}_{i\in\mathbb{N}}$ is non-decreasing from a certain i (see[5]); then it is important to find the integer m determining the largest point at which the sequence decreases, that is, $d_{ORD}(C_i) = \nu(s_{i+1})$ for $i \geq m$. A first approach to this problem can be found in [1], where the author gave the value of m for acute semigroups recalled in (2.1). In [4] (see Theorem2.8 below), we improved this result: by introducing the new parameter

$$t := min\{j \in \mathbb{N} \ \ such\ that \ \ d - j \in S, \ \ d - \ell - j \notin S\},$$

we deeply studied m for $t \leq 4$. In particular we characterized the semigroups

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having $m = 2d - t - g$; in addition we proved that in all the remaining cases $m \leq 2d - 4 - g$.

In the present paper we further develope this topic. In Sections 2 and 3, after fixing the setting and notation, we recall some known results and prove some technical statements. In Section 4 we give exact evaluations or better bounds for m in the unsolved cases. In fact we prove the following facts:

- when $d' \leq d - t \leq d$, then $s_m = 2d - t$ (4.1) and (4.2).
- when $2d' - d \leq d - t < d'$, then $s_m \leq 2d - t$;
  further we give necessary and sufficient conditions on S in order to have $s_m = 2d - t$ or $2d' \leq s_m < 2d - t$ (4.4.1) and (4.4.2).
- when $d - t < 2d' - d$, then $s_m \leq 2d'$;
  in addition we find the necessary and sufficient conditions on S for having $s_m = 2d'$ (4.4.3). We also calculate s_m under particular assumptions (4.6.1).

Finally, in Section 5 we study completely the case $\ell = 2$: we evaluate m in function of the invariant t (see Theorem 5.5) and, as a corollary, we determine m for semigroups of Cohen-Macaulay type three (see Corollary 5.9) as well as for semigroups with multiplicity ≤ 6 (see Corollary 5.6) .

2 Preliminaries

We begin by giving the setting of the paper.

Setting 2.1 In all the article we shall use the following notation. Let $\mathbb{N}$ denote the set of all nonnegative integers. A *numerical semigroup* is a subset S of $\mathbb{N}$ containing 0, closed under summation and with finite complement in $\mathbb{N}$.
We denote the elements of S by $\{s_i\}_{i \in \mathbb{N}}$, with $s_0 = 0 < s_1 < ... < s_i < s_{i+1}....$
We set $S(1) := \{b \in \mathbb{N} \mid b + (S \setminus \{0\}) \subseteq S\}$

We list below the invariants related to a semigroup S we shall use in the sequel.

c := $min\ \{r \in S \mid r + \mathbb{N} \subseteq S\}$, the *conductor* of S
d := the greatest element in S preceding c, *the dominant* of S
c' := $max\{s_i \in S \mid s_i \leq d \ and \ s_i - 1 \notin S\}$, the *subconductor* of S
d' := the greatest element in S preceding c', when $d > 0$
ℓ := $c - 1 - d$, the number of gaps of S greater than d
g := $\#(\mathbb{N} \setminus S)$, the *genus* of S (= the number of gaps of S)
τ := $\#(S(1) \setminus S)$, the *Cohen–Macaulay type* of S
e := s_1, the *multiplicity* of S.

We shall always assume $e > 1$, so that $S \neq \mathbb{N}$. With this notation the semigroup has the following shape (where "$*$" denote gaps and "$\longleftrightarrow$" intervals without any gap):

$$S = \{0, * \ldots *,\ e,\ \ldots,\ d', * \ldots *,\ c' \longleftrightarrow d, \overset{\ell \text{ gaps}}{* \ldots *},\ c \rightarrow\}.$$

Recall also that a semigroup S is called

- *ordinary* if $S = \{0\} \cup \{n \in \mathbb{N},\ n \geq c\}$ [1, Def. 5.1];
- *acute* if either S is ordinary, or S is non-ordinary and c, d, c', d' satisfy $c - d \leq c' - d'$ [1, Def. 5.6.].

According to [3] and [1], for $s_i \in S$ we shall denote

- $N(s_i) := \{(s_j, s_k) \in S^2 \mid s_i = s_j + s_k\} = \{(s_i - s_j,\ s_j) \in S^2\}$
- $\nu(s_i) := \#N(s_i)$, the cardinality of $N(s_i)$
- $d_{ORD}(i) := min\{\nu(s_j) \mid j > i\}$, the *order bound.*

Now we recall some definition and former results for completeness.

Definition 2.2 *We define the parameters m and t as follows*
$m := min\{j \in \mathbb{N}$ *such that the sequence* $\{\nu(s_i)\}_{i \in \mathbb{N}}$ *is non-decreasing for* $i > j\}$
$t := min\{j \in \mathbb{N}$ *such that* $d - j \in S$ *and* $d - \ell - j \notin S\}$.

Theorem 2.3 *Let S be as in Setting 2.1, and let $i \in \mathbb{N}$. Then*

(1) $\nu(s_i) = i + 1 - g$, *for every* $s_i \geq 2c - 1$. ([5, Th. 3.8])

(2) $\nu(s_{i+1}) \geq \nu(s_i)$, *for every* $s_i \geq 2d + 1$. [4, Prop. 3.9.1]

(3) *If S is an ordinary semigroup, then* $m = 0$. ([1, Th. 7.3])

(4) *If S is an acute semigroup then* $c + c' - 2 \leq 2d$ *or* $t = 0$. ([4, Prop. 3.4]).

Remark 2.4 (1). Observe that for each $s_i \geq c$, one has $i = s_i - g$; this equality is no more true if $s_i < c$, hence to simplify the notations our statements shall often deal with s_m instead of m.
(2). Theorem 2.3 implies that $0 < s_m \leq 2d$ for every non-ordinary semigroup. Recalling the definition of $d_{ORD}(i)$ above, one has:

$$d_{ORD}(i) = \nu(s_{i+1}), \quad for \ \ every \ \ i \geq m.$$

The meaning of t will be clear in next Theorem 2.8 where we gather the known results on this argument (see[4, Th. 3.1]). We state beforehand a proposition which allows to re-write in a better way the above cited Theorem 3.1 of [4]. In particular we show when $d - \ell - t$ attains the maximal value $c' - 1$ (it is clear that $d - \ell - t \leq c' - 1$ since $d - \ell - t \notin S$, by definition).

Proposition 2.5 *The following conditions are equivalent:*

(1) $c + c' - 2 \leq 2d$.

(2) $d - \ell - t = c' - 1$.

(3) $c + c' - 2 = 2d - t$.

If these conditions are satisfied then $c' \leq d - t \leq d$.

Proof. (1) $\Longrightarrow$ (2), Let $t' := 2d - (c + c' - 2)$, ($t' \geq 0$ by assumption). Hence $d - \ell - c' + 1 = t'$; and so $d \geq d - t' = c' + \ell - 1 \geq c'$. Then $d - t' \in S$. Moreover $d - \ell - t' = c' - 1 \notin S$, hence $t' \geq t$ (2.2). If $t' > t$, one has $d - \ell - t > d - \ell - t' = c' - 1$, then $d - \ell - t > d$, impossible. It follows $t' = t$, i.e. $d - \ell - t = c' - 1$.
(2) $\Longrightarrow$ 1). If (2) holds, since $c = d + \ell + 1$, then $c + c' - 2 - 2d = d + \ell + 1 + d - \ell - t - 1 - 2d = -t \leq 0$.
(3) $\Longleftrightarrow$ (2) is obvious recalling that $c = d + \ell + 1$.
Finally, if (1),(2),(3) hold, then $d - t = c' + \ell - 1$, and so $c' \leq d - t \leq d$. $\diamond$

Example 2.6 The vice-versa of last statement in (2.5) doesn't hold in general: in the following semigroup we have $c' \leq d - t \leq d$, but $c + c' - 2 > 2d$.
$S = \{0, 25_e, 26, 27, 28, 29_{d'}, 32_{c'}, 33, 34, 35, 36, 37, 38_d, 48_c \rightarrow\}$.
Here: $\ell = 9$, $d - t = 33$, $t = 5$, $c + c' - 2 = 78 > 2d = 76$.
(Further $s_m = 71 = 2d - t$ according to next Theorem 4.1).

The following lemma gives some relations on the elements of S and shows that, under suitable assumptions, certain elements can be greater than the conductor.

Lemma 2.7 *Let S be as in* (2.1) *and let $s \in S$. The following facts hold.*

(1) $d - c' \leq e - \ell - 1$, $c - c' \leq e$, $c' - d' \leq e$, $\ell \leq e - 1$.

(2) $c + c' - 2 > 2d \iff d - c' \leq \ell - 2$.

(3) *Let $c + c' - 2 > 2d$. We have: $d - c' \leq e - 3$ and if S is non-acute, then:*

- (*a*) $2d' \geq c$.
- (*b*) $d - t < d' \Longrightarrow 2d' - \ell \geq c$.

(4) (*c*) $s + 1 - c = s - \ell - d$.
- (*d*) *If* $d - t < s \leq d$, *then:* $s - \ell \in S$.
- (*e*) *If* $s - d \in S$ *and* $d - t < s - d \leq d$, *then* $s \geq c$ *and* $s + 1 - c \in S$.

(5) *Assume S non-ordinary, then:*

- (*f*) $2c' \geq c$ *and either* $d' = 0$ *(i.e.* $c' = e$*) and the equality holds (also, S is acute), or* $d' > 0$ *and* $2c' \geq c + 2$.
- (*g*) $d - t \geq e$.

Proof. (1) We have $d - c' \leq e - 1$, otherwise $d - c' \geq e$ would imply $c' = c$; consequently $c' + e \geq c$ since $c' + e \in S$. Now to get the first inequality, write $c' + e \geq d + \ell + 1$. Further $c' - d' \leq e$, otherwise $d' < d' + e < c'$, with $d' + e \in S$, impossible by definition of d'.
Finally $\ell < e$ because $d + \ell = c - 1$ and $d + e \geq c$ (in fact $d + e \in S$).
(2) follows from (1), by substituting $c = d + \ell + 1$.
(3). When $c + c' - 2 > 2d$, by (2), (1) we have $d - c' \leq \ell - 2$, $\ell \leq e - 1$, hence $d - c' \leq e - 3$.

(a). Since $c' - d' \le e$, by (1), and $d - c' \le e - 3$, as just proved, we get:

$$(*) \qquad d - d' \le 2e - 3.$$

Now suppose S is non-acute. Note that obviously one has

either $d' = 0$, or $c' \le 2d' \le d$, or $2d' \ge c$.

If $d' = 0$, then $c' = e$ and so S is acute, (because $\ell \le e - 1$, by (1)) against the assumption. If the second case holds, let $d' = pe + h$, $p \ge 1$, $0 \le h < e$. By $(*)$ we have: $pe + h = 2d' - d' \le d - d' \le 2e - 3$. Then $p = 1$ and by (1) it follows that

$$c' \le 2d' = 2e + 2h \le d < c \le c' + e,$$

therefore $\qquad e \ge c - c' = d + \ell + 1 - c' \ge \ell + 1 + 2e + 2h - c'$.

From this last chain and the assumption S non-acute we obtain a contradiction:

$$\begin{cases} e - h - (2e - c') \ge \ell + 1 + h \\ e - h - (2e - c') = c' - d' \le \ell. \end{cases}$$

It follows that $2d' \ge c$.

(b). When $d - t < d'$, we have $d' - \ell \in S$, by definition of t (2.2). Thus, $2d' - \ell = d' + (d' - \ell) \in S$; further by (a), we have $2d' \ge c$ and so $2d' - \ell \ge c - \ell = d + 1$. It follows $2d' - \ell \ge c$.

(4). (c), (d) are clear by the equality $c = d + \ell + 1$ and by the definition of t.

To see (e): by (c), $s + 1 - c = s - \ell - d$, then apply (d) to $s - d$.

(5). (f). If S is non-ordinary we have $c' \ge e$, thus $2c' \ge c' + e \ge c$, by (1). Now, if $d' \ge e$, then $c' \ge e + 2$ and so $2c' = c' + c' \ge c' + e + 2 \ge c + 2$.

(g). If $d - t = 0$, then either $c = e$ and S is ordinary, or $d - t < e \le d$, so that $e - \ell \in S$, by definition of t (2.2), impossible by item (1). $\diamond$

Theorem 2.8 [4, Th. 3.1] *With Setting* 2.1, *let S be a non-ordinary semigroup. Let m, t be as in* (2.2). *Then*

(1) *If either* $c + c' - 2 > 2d$ *and* $0 \le t \le 2$, *or* $c + c' - 2 \le 2d$, *then* $m = 2d - g - t$.

(2) *If* $t = 3$ *or* $t = 4$, *then* $m \le 2d - g - t$ *and*
$m = 2d - g - t \iff \{d - 1, ..., d - t + 1\} \cap S \ne \{d - t + 1\}$.

(3) *If* $t \ge 5$ *then* $m \le 2d - g - 4$. *The equality holds if and only if*
$\{d - 1, d - 2, d - 3\} \cap S = \{d - 2\}$ *and* $(d - 4 \in S \iff d - \ell - 4 \in S)$.

Proof. If $c + c' - 2 \le 2d$, we have $c + c' - 2 = 2d - t$ by (2.5), and the equality $m = 2d - t - g$, by ([4, Th.3.1.1]). The other cases are proved in [4, Th.3.1]. $\diamond$

3 Preliminary results.

In order to improve the results of theorems (2.3) and (2.8), we shall analyze the sets $N(s_i)$ (see (2.1)) in detail. Since either for ordinary semigroups or for elements greater then $2c$ all is known, in what follows we shall always assume S non-ordinary and consider $N(s_i)$, only for elements $s_i \le 2c - 1$. First we introduce some new notation and prove some technical facts.

Setting 3.1 Let $s_i \in S$ and let $N(s_i) = \{(x, y) \in S^2 \mid x + y = s_i\}$ as in (2.1):

- $S' := \{r \in S \mid r < c'\} = [0, d'] \cap S$.

- $A(s_i) := \{(x,y),(y,x) \in N(s_i) \mid x < c',\ c' \leq y \leq d\}$;

  further denote by $\alpha(s_i) := \#A(s_{i+1}) - \#A(s_i)$.

- $B(s_i) := \{(x,y) \in N(s_i) \mid (x,y) \in [c',d]^2\ \}$; $\beta(s_i) := \#B(s_{i+1}) - \#B(s_i)$.

- $C(s_i) := \{(x,y) \in N(s_i) \mid (x,y) \in [0,d']^2\ \}$; $\gamma(s_i) := \#C(s_{i+1}) - \#C(s_i)$.

- $A_c(s_i) := \{(x,y),\ (y,x) \in N(s_i) \mid\ x \geq c\}$; $\delta(s_i) := \#A_c(s_{i+1}) - \#A_c(s_i)$.

When $s_{i+1} = s_i + 1$ (e.g, for $s_i \geq c$), we shall often omit indexes, as well we shall write respectively $\alpha, \beta, \gamma, \delta$ when no confusion arises.

Remark 3.2 (1) With the above Notation 3.1, we obtain

$$N(s_i) = A(s_i) \cup B(s_i) \cup C(s_i) \cup A_c(s_i)$$

where the union is disjoint. Therefore to calculate $\nu(s_{i+1}) - \nu(s_i)$ we shall use the equality: $\nu(s_{i+1}) - \nu(s_i) = \alpha(s_i) + \beta(s_i) + \gamma(s_i) + \delta(s_i)$.
As we shall prove later, the above summands can be easily known for each element $s_i \in S$, with the exception of $\gamma(s_i)$; in fact the subsets $C(s_i)$ are quite difficult to manage if $s_i < 2d'$. For this reason, when $s_i < 2d'$ we can evaluate $\nu(s_{i+1}) - \nu(s_i)$ only in particular cases; on the other hand we are able to calculate $\nu(s_{i+1}) - \nu(s_i)$ for $s_i \geq 2d'$.
(2) For the pair $(0, s_i)$ of $N(s_i)$, note that: $(0, s_i) \in A_c(s_i)$ if $s_i \geq c$, while $(0, s_i) \in A(s_i)$ if $c' \leq s_i \leq d$, and $(0, s_i) \in C(s_i)$ in the remaining cases ($s_i \leq d'$).

Lemma 3.3 *Let* $s_i \in S$ *and let* $A(s_i)$, $\alpha(s_i)$ *be as in Setting 3.1. Then:*

(1) *If either* $s_i < c'$, *or* $s_i > d + d'$, *then* $A(s_i) = \emptyset$.

$$(2)\ \alpha(s_i) = \left[\begin{array}{lll} -2 & if & (s_{i+1} - c' \notin S' \ and \ s_i - d \in S') \\ 0 & if & \left[\begin{array}{ll} either & (s_{i+1} - c' \notin S' \ and \ s_i - d \notin S') \\ or & (s_{i+1} - c' \in S' \ and \ s_i - d \in S') \end{array}\right. \\ 2 & if & (s_{i+1} - c' \in S' \ and \ s_i - d \notin S'). \end{array}\right.$$

Proof. First note that for each $s \in S$ and for each $(x,y) \in A(s)$ we have $x \neq y$ because $A(s) = \{(x,y),(y,x) \in S^2 \mid x + y = s,\ x \leq d',\ c' \leq y \leq d\}$.
(1). If $s_i \leq c' - 1$, $A(s_i) = \emptyset$, by definition. If $x + y = s_i > d + d'$, and $x \leq d'$, then $y > d$. Hence $s_i > d + d'$, implies $A(s_i) = \emptyset$.
(2). *Case (a)* - If $c' = d$, then (2) holds because we have

$$A(s_i) \subseteq \{(s_i - d, d), (d, s_i - d)\}, \quad A(s_{i+1}) \subseteq \{(s_{i+1} - d, d), (d, s_{i+1} - d)\}.$$

Case (b) - If $c' \leq d - 1$, denote by $i_0 \in \mathbb{N}$ the index such that $s_{i_0} = c'$. We divide the proof in several subcases.

- If $i \leq i_0 - 2$, then $A(s_i) = A(s_{i+1}) = \emptyset$ and $\alpha(s_i) = 0$, so (2) holds because we have $s_{i+1} - c' \notin S', s_i - d \notin S'$.

- If $i = i_0 - 1$, then $A(s_i) = \emptyset$, $A(s_{i+1}) = A(c') = \{(0, c'), (c', 0)\}$ and $\alpha(s_i) = 2$, so (2) holds because $s_{i+1} - c' = 0 \in S'$, $s_i - d \notin S'$.

- If $c' \leq s_i \leq d - 1$, then $s_{i+1} = s_i + 1$, hence for each $(x,y) \in A(s_i)$, with $c' \leq y \leq d - 1$, one has: $(x,y) \in A(s_i) \Longleftrightarrow (x, y+1) \in A(s_{i+1})$. Now call $A' := \{(x,y),(y,x) \in A(s_i) \mid c' \leq y \leq d - 1\}$,

$A'' := \{(x, y+1), (y+1, x) \mid (x,y) \in A'\}$. Clearly $\#A' = \#A''$, further we have

$$A(s_i) = \left[\begin{array}{ll} A' \cup \{(s_i - d, d), (d, s_i - d)\} & if \quad s_i - d \in S' \\ A' & if \quad s_i - d \notin S' \end{array} \right.$$

$$A(s_{i+1}) = \left[\begin{array}{ll} A'' \cup \{(s_{i+1} - c', c'),\ (c', s_{i+1} - c')\} & if \quad s_{i+1} - c' \in S' \\ A'' & if \quad s_{i+1} - c' \notin S' \end{array} \right. .$$

Then (2) is true.

- If $s_i = d$, then $s_{i+1} = c$, $A(s_i) = \{0, d), (d, 0)\}$ and we are done since from the inequalities $0 < \ell + 1 = c - d \le c - c' \le e$ (2.7.1), we deduce that

$$A(s_{i+1}) = \left[\begin{array}{ll} \emptyset & if \ \ c' \neq c - e \\ \{(c', e)(e, c')\} & if \ \ c' = c - e. \end{array} \right.$$

- If $c \le s_i \le d + d' - 1$, then proceed as in case $c' \le s_i \le d - 1$.

- If $s_i = d + d'$, then $A(s_i) = \{(d', d), (d, d')\}$, $A(s_{i+1}) = \emptyset$, by (1), so that $\alpha(s_i) = 2$ and assertion (2) still holds because $d + d' + 1 - c' \notin S'$ (see (1)).

- If $s_i > d + d'$, then $A(s_i) = A(s_{i+1}) = \emptyset$, $\alpha(s_i) = 0$ and (2) is satisfyed since $s_i - d \notin S'$ and $s_{i+1} - c' \notin S'$ (see (1). $\diamond$

Lemma 3.4 *For $s_i \in S$, let $B(s_i)$ and $\beta(s_i)$ be as in* (3.1) *and let $i_1 \in \mathbb{N}$ be such that $s_{i_1} = 2c'$.*

(1) *If $d' > 0$, then $s_{i_1 - 2} = 2c' - 2$.*

(2) *If $s_i < 2c'$, or $s_i > 2d$, then $B(s_i) = \emptyset$.*

(3) $\beta(s_i) = \left[\begin{array}{lll} 0 & if & s_i \le s_{i_1 - 2} \ \ or \ \ s_i > 2d \\ 1 & if & s_{i_1 - 1} \le s_i \le c' + d - 1 \\ -1 & if & c' + d \le s_i \le 2d. \end{array} \right.$

Proof. (1) follows by (2.7.5).

(2). Since by definition $B(s_i) \subseteq [c', .., d]^2$, obviously for $s_i < 2c'$, or $s_i > 2d$ one has $B(s_i) = \emptyset$.

(3). The first case is obvious by (2).

- For the case $i = i_1 - 1$, i.e. $s_{i+1} = 2c'$, clearly we have $B(s_i) = \emptyset$, while $B(2c') = \{(c', c')\}$. Thus $\beta = 1$.

- If $2c' \le s_i \le c' + d - 1$, recall that $2c' \ge c$ for every non-ordinary semigroup by (2.7.4). Therefore if $s_i \ge 2c'$, then $s_i + 1 = s_{i+1}$. Now let $s_i = 2c' + h$, with $0 \le h \le d - c' - 1$. Then:

$$B(s_i) = \{(c', c' + h), (c' + 1, c' + h - 1), ..., (c' + h, c')\}$$

and so $\#B(s_i) = h+1$, $\#B(s_i+1) = h+2$, $\beta(s_i) = \#B(s_i+1) - \#B(s_i) = 1$.

- If $c' + d \le s_i \le 2d$, let $s_i = 2d - k$, with $0 \le k \le d - c'$. Then

$$B(s_i) = \{(d - k, d), (d - k + 1, d - 1), ..., (d, d - k)\}$$

and so, $\#B(s_i) = k + 1$, $\#B(s_i + 1) = k$ (in particular for $k = 0$, note that $B(s_i) = \{(d, d)\}$, $B(s_i+1) = \emptyset$). Hence $\beta(s_i) = \#B(s_i+1) - \#B(s_i) = -1$. $\diamond$

By the definition of $C(s)$ one immediately obtains the following equalities.

Lemma 3.5 *Let $s \in S$ and let $C(s)$ be as in* (3.1). *We have:*

(1) *If* $s \geq 2d' + 1$, *then* $C(s) = \emptyset$ *and* $\gamma(s) = 0$.

(2) $C(2d') = \{(d', d')\}$ *and* $\gamma(2d') = -1$.

Example 3.6 This example shows that for $s < 2d'$ we can have $C(s) \neq \emptyset$. Let $S = \{0, 10_e, 11, 12, 13_{d'}, 15_d, 20_c \rightarrow\}$ For $s = 2d' - 4 = 22$, we have $C(s) = \{(10, 12), (11, 11), (12, 10)\}$.

Lemma 3.7 *Let* $s_i \in S$ *and let* $A_c(s_i)$, $\delta(s_i)$ *be as in Setting 3.1. Then:*

(1) $A_c(s_i) = \emptyset \iff s_i < c$.

(2)
$$\delta(s_i) = \left[\begin{array}{ll} if \quad 0 \leq s_i \leq 2c - 1 : & \left[\begin{array}{lll} 0 & \iff & s_{i+1} - c \notin S \\ 2 & \iff & s_{i+1} - c \in S \end{array}\right. \\ if \quad s_i \geq 2c : & \left[\begin{array}{l} 1 \quad . \end{array}\right. \end{array}\right.$$

Proof. (1) is obvious: if $s_i = c + h \geq c$, then $(s_i, 0) \in A_c(s_i)$.
(2). For $s_i < d$, we have $s_{i+1} - c \notin S$, and we are done. If $s_i = d$, then $s_{i+1} = c$, hence $A_c(d) = \emptyset$ by (1), $A_c(c) = \{(0, c), (c, 0)\}$.
For $d < s_i \leq 2c - 1$, since $s_i \geq c$, we have $s_{i+1} = s_i + 1$, thus let $s := s_i$ and let $X(s) := \{(x, y) \in A_c(s) \mid x \geq c\} \subseteq A_c(s)$: we write $X(s) + (1, 0)$ in order to mean the set $\{(x, y) + (1, 0) \mid (x, y) \in X(s)\}$. The statement follows from the inclusions

$$X(s) + (1, 0) \subseteq X(s + 1) \subseteq (X(s) + (1, 0)) \cup \{(c, s + 1 - c)\}.$$

In fact for each pair $(c + h, y) \in X(s)$, one has $(c + h + 1, y) \in X(s + 1)$. Further for each pair $(x, y) \in X(s)$ one has $x \neq y$, otherwise $x = c + h = y = s - c - h$ would imply $s = 2c + 2h > 2c$, which contradicts the assumption $s \leq 2c - 1$. Hence $\#A_c(s) = 2(\#X(s)) = 2\#(X(s) + (1, 0))$ and (2) holds.
When $s_i \geq 2c$ the result follows by a direct computation, because in this case $N(s_i) = A_c(S_i)$. $\diamond$

Remark 3.8 Since we shall use deeply the preceding lemmas (3.3),(3.4),(3.5), it is convenient to note that in the case $s \leq 2d < c + c' - 2$, since $d - \ell \leq c' - 2$ (2.7.2), we have $s + 1 - c \in S \iff s + 1 - c \in S'$. In fact $s + 1 - c = s - \ell - d \leq d - \ell$. Moreover for $s \leq 2c' - 2$ we have $s + 1 - c' \in S \iff s + 1 - c' \in S'$ and the same holds for $s - d$.

As follows by lemmas (3.3),(3.4),(3.5),(3.7) and Remark 3.2, for each element $s_i \in S$ the difference $\nu(s_{i+1}) - \nu(s_i)$ can be easily described in function of γ. This is shown in next Theorem (3.10), by means of a series of tables.

Setting 3.9 Let $S' = \{s \in S \mid s \leq d'\}$ as in (3.1). In the following tables for an integer r we write respectively " $\times$ " if $r \in S'$, " $\bigcirc$ " if $r \notin S'$.

Theorem 3.10 *With setting* (3.1) *and* (3.9), *let* $i_1 \in \mathbb{N}$ *be such that* $s_{i_1} = 2c'$. *The following tables describe the difference* $\nu(s_{i+1}) - \nu(s_i)$ *for* $s_i \in S$, $s_i < 2c$.

(a) *If* $i \le i_1 - 2$ *(hence* $s_i \le 2c' - 2$*), then* $\beta = 0$*,*

$$\begin{bmatrix} s_{i+1} - c & s_i - d & s_{i+1} - c' & \alpha & \beta & \delta & \nu(s_{i+1}) - \nu(s_i) \\ \notin S & \bigcirc & \bigcirc & 0 & 0 & 0 & \gamma \\ \notin S & \times & \bigcirc & -2 & 0 & 0 & \gamma - 2 \\ \notin S & \bigcirc & \times & 2 & 0 & 0 & \gamma + 2 \\ \notin S & \times & \times & 0 & 0 & 0 & \gamma \\ \in S & \bigcirc & \bigcirc & 0 & 0 & 2 & \gamma + 2 \\ \in S & \bigcirc & \times & 2 & 0 & 2 & \gamma + 4 \\ \in S & \times & \bigcirc & -2 & 0 & 2 & \gamma \\ \in S & \times & \times & 0 & 0 & 2 & \gamma + 2 \end{bmatrix}.$$

(b) *If* $s_i \in [s_{i_1-1}, c' + d - 1] \cap S$*, then* $s_{i+1} - c' \notin S'$*,* $\beta = 1$*,*

$$\begin{bmatrix} s_{i+1} - c & s_i - d & s_{i+1} - c' & \alpha & \beta & \delta & \nu(s_{i+1}) - \nu(s_i) \\ \in S & \bigcirc & \bigcirc & 0 & 1 & 2 & \gamma + 3 \\ \in S & \times & \bigcirc & -2 & 1 & 2 & \gamma + 1 \\ \notin S & \bigcirc & \bigcirc & 0 & 1 & 0 & \gamma + 1 \\ \notin S & \times & \bigcirc & -2 & 1 & 0 & \gamma - 1 \end{bmatrix}.$$

(c) *If* $s_i \in [\, c' + d, 2d\,] \cap S$*, then* $s_i + 1 \in S$*,* $s_i - d \notin S'$*,* $s_i + 1 - c' \notin S'$*,*

$$\begin{bmatrix} s_i + 1 - c & s_i - d & s_i + 1 - c' & \alpha & \beta & \gamma & \delta & \nu(s_i+1) - \nu(s_i) \\ \in S & \bigcirc & \bigcirc & 0 & -1 & 0 & 2 & 1 \\ \notin S & \bigcirc & \bigcirc & 0 & -1 & 0 & 0 & -1 \end{bmatrix}.$$

(d) *If* $s_i \in [2d+1, 2c-1\,] \cap S$*, then* $s_i + 1 \in S$*,* $s_i - d \notin S'$*,* $s_i + 1 - c' \notin S'$*,*

$$\begin{bmatrix} s_i + 1 - c & s_i - d & s + 1 - c' & \alpha & \beta & \gamma & \delta & \nu(s_i+1) - \nu(s_i) \\ \in S & \bigcirc & \bigcirc & 0 & 0 & 0 & 2 & 2 \\ \notin S & \bigcirc & \bigcirc & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof. It follows by ((3.3),(3.4),(3.5),(3.7)). In case (b), we have $s_i + 1 \in S$ by (2.7.4) (recall: in this section we assume S non ordinary). In cases $(c), (d)$ one has $s_i - d \notin S'$, because $s_i - d \ge c'$. $\diamond$

For $s_i \ge c' + d$ we know $\nu(s_{i+1}) - \nu(s_i)$ by Theorem 2.3 and by items (c),(d) of Theorem 3.10 above. Now we shall achieve the answer for $2d' \le s_i < c' + d$; when $2d' + 1 \le s_i < c' + d$, since $0 \le \beta(s_i) \le 1$, $\gamma = 0$, it is convenient to express the difference $\nu(s_{i+1}) - \nu(s_i)$ in function of the parameter β.

Theorem 3.11 *With setting* (2.1), (3.1), (3.9)*,let* S *be a non-ordinary semigroup and let* $s_i \in [2d', c' + d - 1] \cap S$*. Then the difference* $\nu(s_{i+1}) - \nu(s_i)$ *can be evaluated as follows.*

(1) *Let* $s_i = 2d'$*. Then:* $\beta = 0$*,* $\gamma = -1$ *and*

$s_{i+1}-c$	s_i-d	$s_{i+1}-c'$	α	β	γ	δ	$\nu(s_{i+1})-\nu(s_i)$
$\notin S$	$\times$	$\bigcirc$	-2	0	-1	0	-3
$\notin S$	$\times$	$\times$	0	0	-1	0	-1
$\notin S$	$\bigcirc$	$\bigcirc$	0	0	-1	0	-1
$\in S$	$\times$	$\bigcirc$	-2	0	-1	2	-1
$\notin S$	$\bigcirc$	$\times$	2	0	-1	0	1
$\in S$	$\times$	$\times$	0	0	-1	2	1
$\in S$	$\bigcirc$	$\bigcirc$	0	0	-1	2	1
$\in S$	$\bigcirc$	$\times$	2	0	-1	2	3

(2) *Assume that* $2d'+1 \le s_i \le c'+d-1$. *Then:* $\beta \in \{0,1\}$, $\gamma = 0$ *and*

$s_{i+1}-c$	s_i-d	$s_{i+1}-c'$	α	γ	δ	$\nu(s_{i+1})-\nu(s_i)$
$\notin S$	$\times$	$\bigcirc$	-2	0	0	$-2+\beta$
$\notin S$	$\times$	$\times$	0	0	0	β
$\notin S$	$\bigcirc$	$\bigcirc$	0	0	0	β
$\notin S$	$\bigcirc$	$\times$	2	0	0	$2+\beta$
$\in S$	$\times$	$\bigcirc$	-2	0	2	β
$\in S$	$\times$	$\times$	0	0	2	$2+\beta$
$\in S$	$\bigcirc$	$\bigcirc$	0	0	2	$2+\beta$
$\in S$	$\bigcirc$	$\times$	2	0	2	$4+\beta$

Proof. The theorem follows by (3.2),(3.3),(3.4),(3.5),(3.7) and Th.3.10. $\diamond$

4 New evaluations or bounds for m.

By Theorem 2.8 we know that $m = 2d - t - g$ under suitable conditions, but this equality is not true in general. However when $d - t \ge d'$, we always have $m = 2d - t - g$: this is proved by the following theorems (4.1) and (4.2).

4.1 The case $\mathbf{d - t \ge d'}$.

Theorem 4.1 *Let t, m be as in (2.2). If* $c' \le d - t \le d$, *then* $s_m = 2d - t$.

Proof. Let $s = 2d - t + h$, with $0 \le h \le t$. In this case $c' + d \le s \le 2d$ and so $s+1 \in S$. Further $s+1-c \in S \iff h \ge 1$ by the definition of t and by (2.7.4). Then we have the result by (2.3) and by table 3.10.(c). $\diamond$

Theorem 4.2 *Assume $d-t \le d'$. Then $d' > 0$ and the following relations hold*

(1) $s_m \le d + d'$.

(2) $s_m = d + d' \iff d - t = d'$.

Proof. First note that $d - t \ge e$ (2.7.5), hence $d' \ge d - t > 0$; thus $d + d' \ge c$ and for $s_i \in [d + d', 2d]$, we have $s_i + 1 \in S$.
(1). By (2.3.2) it suffices to prove that for every $s_i \in [d + d' + 1, 2d]$ we have:

$$\nu(s_{i+1}) \ge \nu(s_i).$$

- For $d+d'+1 \leq s_i \leq c'+d-1$, we achieve the proof by using table 3.11.(2). In fact we have $\alpha = 0$, $\beta \geq 0$, by (3.3), (3.4) and so we are done.

- For $c' + d \leq s_i \leq 2d$, we get $\nu(s_{i+1}) = \nu(s_i) + 1$ by table 3.10.(c): in fact $s_i - d \in S$ (since $c' \leq s_i - d \leq d$) and the assumption $d - t \leq d'$ assures that $d - t < s_i - d \leq d$. Therefore $s_{i+1} - c \in S$, by (2.7.4).

(2) Let $s = d+d'$; we shall prove that $\begin{cases} \nu(d+d'+1) < \nu(s), & if \quad d-t = d' \\ \nu(d+d'+1) \geq \nu(s), & if \quad d-t < d'. \end{cases}$

Clearly we have $2d'+1 < s < c'+d-1$ and $s-d \in S'$; further $s+1-c = d'-\ell$, so: $s+1-c \notin S \Longleftrightarrow d-t = d'$ (by (2.2) and (2.7.4)).

Finally, $s+1-c' \notin S'$, because $s+1-c' \geq d'+1$; hence we are in the first or in the fifth row of table 3.11.(2) and we are done because $\beta \in \{0, 1\}$. $\diamond$

4.2 The case $\mathbf{d - t < d'}$.

Now we shall assume $d - t < d'$: we already know that $s_m < d + d'$ (Th. 4.2). In this case we can have $s_m \neq 2d - t$. For example, if $t = 4$ and $\{d-1, d-2, d-3\} \cap S = \{d-3\}$, one has: $d - t = d - 4$, $d' = d - 3$ and $m < 2d - g - t$, by (2.8.2). This example can be generalized: in fact we shall prove that $m < 2d-t-g$ whenever $2d'-d < d-t < d'$, $c' = d$ and $d-t+1 \in S$ (4.6). We shall estimate the difference $\nu(s_{i+1}) - \nu(s_i)$ for each $s_i \geq 2d'$ by using the tables $(1), (2)$ of (3.11). Note that in the case $d - t < d'$, we have $c + c' - 2 > 2d$ by (2.5), moreover we have $t \geq 3$ and so S is non-acute (2.3.4). Therefore, by Lemma 2.7.3, $2d' \geq c$, and for each $s \in S$, s greater or equal to $2d'$, we have $s+1 \in S$. At first (3.10) and (3.11.2) give easily the next corollary.

Corollary 4.3 (1) *Let $s_i \in S$, and let $2d' + 1 \leq s_i \leq c' + d - 1$. The following conditions are equivalent:*

(a) $\nu(s_{i+1}) < \nu(s_i)$.

(b) $s_{i+1} - c \notin S$, $s_i - d \in S'$, $s_{i+1} - c' \notin S'$.

(2) *If $s_i \in S$, $s_i \geq 2d' + 1$ and $s_{i+1} - c \in S$, then $\nu(s_i) \leq \nu(s_{i+1})$.*

Proof. (1) is immediate by (3.11.2), since we have $\beta \in \{0, 1\}$.
(2) It follows by (1) for $s_i < c' + d$; for $s_i \geq c' + d$, see tables $(c), (d)$ of (3.10) and (2.3.2). $\diamond$

Theorem 4.4 *Assume $d - t < d'$. We have:*

(1) *If $d - t \geq 2d' - d$:*

(a) $s_m = 2d - t$ *if* $\left[\begin{array}{ll} either & d - t = 2d' - d \\ or & d - t > 2d' - d \ \ and \ \ 2d - t + 1 - c' \notin S'. \end{array} \right.$

(b) $s_m < 2d-t$ *in the other cases:* $(2d-t+1-c' \in S'$ *and* $d-t > 2d'-d)$.

(2) *In case* (1.b) *consider the set*
$X := \{s \in S \cap [2d' + 1, 2d - t - 1] \mid s \ verifies \ conditions \ 4.3.(b)\}$. *Then:*

(c) $2d'+1 \le s_m < 2d-t$ *if and only if* $X \ne \emptyset$;
in this case s_m *is the maximum of* X.

(d) $s_m = 2d'$ *if and only if* $X = \emptyset$ *and* $2d'$ *verifies one of the first four rows in table* 3.11.(1).

(e) $s_m < 2d'$ *in the remaining cases.*

(3) *If* $d-t < 2d'-d$:

(f) $s_m \le 2d'$,

(g) $s_m = 2d' \Longleftrightarrow (2d'-d \in S \Longleftrightarrow 2d'+1-c \in S)$ *and* $2d'+1-c' \notin S$.

Proof. (1). When $d-t<d'$, we know that $s_m < d+d'$, by (4.2) . We start by considering the elements $s \in S$ such that $2d-t \le s < d+d'$; thus let

$$s = 2d-t+k, \text{ with } 0 \le k < d'-(d-t) .$$

Suppose $2d-t \ge 2d'$.

- If $k>0$, we claim that $\nu(2d-t+k) \le \nu(2d-t+k+1)$. First notice that
($*$) $2d' \le 2d-t < s < d+d'$ by the assumptions;
($**$) $s-d \in S \Longrightarrow s+1-c \in S$ by (2.7.4); in fact, $d-t < s-d < d$.
Now the claim follows by (4.3.1)

- If $k=0$, i.e. $s=2d-t$, then: $s+1-c \notin S$, $s-d = d-t \in S'$.
If $s > 2d'$, then by (4.3.1), $\nu(s+1) < \nu(s) \Longleftrightarrow s+1-c' \notin S'$.
Then (a) and (b) follow.
If $s = 2d'$, then s verifies one of the first two rows of table 3.11.(1) so that $\nu(s+1) < \nu(s)$.
(2). Suppose $2d' < 2d-t$ and $s_m < 2d-t$. Then: (c) is immediate by (4.3.1); (d) follows from (c) and table 3.11.(1).
(3). Suppose $2d-t < 2d'$. If $2d' \le s < d'+d$, then $d-t < 2d'-d \le s-d < d$; and so $s-d \in S \Longrightarrow s+1-c \in S$ by (2.7.4). Then: if $s > 2d'$, (f) is proved by (4.3); if $s = 2d'$ one obtains (g) by table 3.11.(1). $\diamond$

When $d-t<d'$, Theorem 4.4 gives upper bounds for s_m and the exact value under certain assumptions, in particular when $2d-t = 2d'$. For the remaining cases we obtain more precise estimations of s_m in some special situation (see Corollary 4.6).

Lemma 4.5 *Suppose* $2d-t < 2d'$ $(so, d-t < d'-1)$. *Let* $s \in S$ *be such that* $2d-t+1 \le s < 2d'$ *and let* $C(s)$ *be as in* (3.1). *Then* $s > c$ *and*

(1) *If* $(x,y) \in C(s)$, *then* $x > d-t$, $y > d-t$.

(2) *The following inequalities hold*: $\begin{cases} d-t+1 < s-d' < d' \\ d-t+2 < s+1-d' \le d'. \end{cases}$

(3) *Moreover if* $[d-t+1, d'-1] \cap S = \emptyset$, *then*

$$\gamma(s) = \left[\begin{array}{lll} 0 & if & 2d-t+1 \le s \le 2d'-2 \\ 1 & if & s = 2d'-1. \end{array} \right.$$

Proof. We have $2d-t \in S,\ 2d-t > d$ (by 2.7.5), and so $2d-t \geq c$.
(1) is easy.
(2). We have: $\quad d-t+1 \leq s-d < s-d' < d'$.
(3). Suppose now that $[d-t+1, d'-1] \cap S = \emptyset$ and let $2d-t+1 \leq s \leq 2d'-1$.
Then: $\quad C(s)=\emptyset, \quad C(s+1) = \left[\begin{array}{cc} \emptyset & if \quad s \leq 2d'-2 \\ \{(d',d')\} & if \quad s = 2d'-1 \end{array} \right.$;
in fact, if $(x,y) \in C(s)$ (resp. $C(s+1)$), with $x, y < d'$, then $d-t < x, y < d'$ by (1), impossible by the assumption $[d-t+1, d'-1] \cap S = \emptyset$. Further by (2) and the assumptions we have: $s-d' \notin S'$, and also $s+1-d' \in S' \Longleftrightarrow s = 2d'-1$. Then the result follows. $\diamond$

Corollary 4.6 *Suppose $d-t \leq d'-1$.*

(1) *If* $\ d-t \leq 2d'-d-1 \ $ *and* $\ [d-t+1, d'-1] \cap S = \emptyset$,
then $\quad s_m = \left[\begin{array}{lll} 2d' & if & 2d'+1-c \notin S \\ 2d-t & if & 2d'+1-c \in S. \end{array} \right.$

(2) *If* $d-t > 2d'-d$ *and* $c'=d$, *then* $\left[\begin{array}{lll} s_m = 2d-t & if & d-t+1 \notin S \\ s_m < 2d-t & if & d-t+1 \in S. \end{array} \right.$

Proof. (1). By (4.4.3.f), we know that $s_m \leq 2d'$. Since by the assumptions we have $d-t < 2d'-d < 2d'+1-d \leq 2d'+1-c' < d'$, by (4.4.3$g$) we get $s_m = 2d' \Longleftrightarrow 2d'+1-c \notin S$.
Suppose $s_m < 2d'$: if $s \in S$ and $2d-t \leq s$, then $s \geq c$ (see the proof of (4.5)) and so $s+1 \in S$. Now for $2d-t+1 \leq s < 2d'$, we have $s < 2c'-2$ and so $\nu(s+1) \geq \nu(s)$ by table 3.10.(a), because $s-d \notin S'$ and $\gamma \geq 0$ by Lemma 4.5.3. Now it suffices to show that $\nu(2d-t+1) < \nu(2d-t)$. We have: • $\gamma(2d-t) \leq 1$. To see this fact, note that:
$$C(2d-t+1) = \left[\begin{array}{ll} \emptyset & if\ d-t \neq 2d'-d-1 \\ \{(d',d')\} & if\ d-t = 2d'-d-1. \end{array} \right.$$
In fact, clearly, $(x,y) \in C(2d-t+1) \Longrightarrow x > d-t,\ y > d-t$ (otherwise, $x \leq d-t \Longrightarrow y > d$, impossible); then, $x < d' \Longrightarrow d-t < x < d'$, impossible since $[d-t+1, d'-1] \cap S = \emptyset$. Finally, if $x = d'$, then
$d-t < y = 2d-t+1-d' \leq d'$ (by assumption) and so $y = d',\ 2d-t = 2d'-1$.
Now, for $s := 2d-t$ we have:
$$s+1-c \notin S,\ s-d = d-t \in S',\ s+1-c' \notin S$$
($s+1-c' \notin S$, because $d-t+1 = s+1-d \leq s+1-c' \leq 2d'-c' < d'$). Hence by the second row of table 3.10.(a), and since $\gamma \leq 1$, we get $\nu(s_{i+1}) - \nu(s_i) = \gamma - 2 \leq -1$.
(2). Under these assumptions, $\ 2d-t+1-c' = d-t+1$. Then the result follows by (4.4.1a). $\diamond$

Example 4.7 When either of the assumptions $[d-t+1, d'-1] \cap S = \emptyset$ and $c' = d\ $ does not hold, the statements (1), (2) of (4.6) are not always true, as the following examples show.

(1) Let $S = \{0, 22_e, 27, 28, 30, 31_{d-t}, 32, 33, 35_{d'}, 38_d, 44_c \rightarrow\}$. (Here (4.6.1) fails). We have $69 = 2d-t = 2d'-1$, $[d-t+1, d'-1] \cap S \neq \emptyset$. By (4.4.3) we know that $s_m \leq 2d'-1 = 69 < 2c'-2$. Moreover $\nu(70) > \nu(69)$, by table 3.10.(a). In fact

for $s = 69$ we have: $s+1-c = 26 \notin S,\ s-d = 31 \in S',\ s+1-c' = 32 \in S'$ and $\gamma(s) = 1$ because $C(69) = \emptyset,\ C(70) = \{(35,35)\}$. Hence $s_m < 2d'-1$. One can verify that $s_m = 68$, with $\nu(68) = 8,\ \nu(69) = 6$.

(2). (a) Let $S = \{0, 18_e, 21, 22, 24, 25_{d-t}, 27_{d'}, 29_{c'}, 30_d, 36_c \rightarrow\}$ $(\ell = t = 5)$.
Here: $55 = 2d-t > 2d' = 54,\ c' \neq d,\ d-t+1 \notin S$. Since $2d-t+1-c' = 27 \in S'$, we have $s_m < 2d-t = 55$ by $(4.4.1b)$. One can verify that $s_m = 54 = 2d-t-1$, with $\nu(54) = 9,\ \nu(55) = 6$.

(b) Let $S = \{0, 18_e, 21, 22, 24, 25_{d-t}, 26_{d'}, 29_{c'}, 30_d, 36_c \rightarrow\}$ $(\ell = t = 5)$.
Here: $c' \neq d,\ \ d-t = 25 > 2d'-d = 22,\ d-t+1 \in S$ and $s_m = 2d-t$ by $(4.4.1a)$, with $\nu(55) = 8,\ \ \nu(56) = 6$.

In next corollary we collect all the cases that give $s_m = 2d-t$.

Corollary 4.8 *With setting 2.1, let* t, m *be as in* 2.2. *Then* $s_m = 2d-t$ *in the following cases.*

(1) *If* $d-t \geq d'$.

(2) *For* $2d'-d < d-t < d'$: *if and only if* $2d-t+1-c' \notin S'$.

(3) *If* $2d'-d = d-t\ (< d')$.

(4) *If* $d-t < 2d'-d$ *and* $[d-t+1, d'-1] \cap S = \emptyset$: *if and only if* $2d'+1-c \in S$.

All the statements of (2.8) can now be easily deduced by (4.1),(4.2),(4.4),(4.6). To exemplify, in part (1) of next Corollary we derive explicitely the results in case $t \geq 5$.

Corollary 4.9 *If* $t \geq 5$, *then*

(1) $s_m \leq 2d-4$ *and the equality holds if and only if* $\{d-1, d-2, d-3\} \cap S = \{d-2\}$ *and* $(d-4 \in S \iff d-\ell-4 \in S)$.

(2) *When* $d-t \geq 2d'-d$, *then* $s_m \leq 2d-t$.

(3) *When* $d-t < 2d'-d$, *then* $s_m \leq 2d' \leq 2d-6$.

Proof. If $d' = d-2$, then $c' = d,\ d-t < 2d'-d = d-4$. It follows (4.4.3): $s_m \leq 2d-4$ and $s_m = 2d-4 \iff (d-4 \in S \iff d-\ell-4 \in S)$ and $d-3 \notin S$. Then: $s_m = 2d-4$ if and only if $\{d-1, d-2, d-3\} \cap S = \{d-2\}$ and $(d-4 \in S \iff d-\ell-4 \in S)$.
If $d' \leq d-3$, we have to consider two cases:
when $d-t \geq 2d'-d$, then $s_m \leq 2d-t$ (4.4.1),
when $d-t < 2d'-d$, then $s_m \leq 2d' \leq 2d-6$ (4.4.3).
This proves the corollary. ⋄

Example 4.10 When $t = 3$, or $t = 4$, the cases still unsolved correspond exactly to the situation $[d-t+1, d-1] \cap S = \{d-t+1\}$ already considered in (2.8): for instance let $t = 3$. This condition means $c' = d$ and $d' = d-2$. So we are in case (1.b) of Theorem 4.4. Then by (4.4.2) one can easily verify that

$$s_m = 2d' = 2d - 4 \Longleftrightarrow d - \ell - 4 \notin S \text{ and } d - 4 \in S.$$

In the remaining cases: $s_m \leq 2d - 5$. For ℓ small, we are able to find s_m by a direct computation. We show the results for $t = \ell = 3$.

In this case $d-5 \in S,\ d-6 \notin S$ and

$$\left[\begin{array}{lll} if & d-4 \notin S, & then\ \ s_m = 2d-5 \\ if & d-4 \in S,\ d-\ell-4 = d-7 \notin S, & then\ \ s_m = 2d-4 \\ if & d-4 \in S,\ d-\ell-4 = d-7 \in S, & then \left[\begin{array}{l} d-8 \notin S \Longrightarrow s_m = 2d-5 \\ d-8 \in S \Longrightarrow s_m = 2d-7. \end{array}\right. \end{array}\right.$$

Hence $s_m \geq 2d - 7 = 2d' - \ell$ for each S with $t = \ell = 3$ and the lower bound is achieved by any semigroup such that

$$[d-8, c] \cap S = \{d-8, d-7, d-5, d-4, d-3, d-2, d, d+4 = c\}$$

Also for $t = 3,\ \ell = 4$ one can verify that $s_m \geq 2d - 8 = 2d' - \ell$.

Remark 4.11 In general, we have a feeling that the case $d - t > 2d' - d$, with $c' = d$ and $d - t + 1 \in S$ considered in (4.6.2) is the worst one in order to give a lower bound for s_m. After many calculations we conjecture that in this case we always have $s_m \geq 2d' - \ell$ (recall that if $s_m \neq 2d - t$, then $d - t < d'$ by (4.8) and so $2d' - \ell \in S$ by Prop. 2.7.3). We illustrate the situation with an example which shows that we can have $s_m << 2d - t$.

$S = \{0, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42, 43, 44, 47_{d-t}, 48_{d'}, 50_d, 61_c \rightarrow\}$

$(\ell = 10,\ t = 3,\ 2d - t = 97)$. One can check that $s_m = 88 = 2d - t - 9 < 2d' = 96$, and $\nu(88) = 9,\ \ \nu(89) = 8$. Note that in this case we have $s_m > 2d' - \ell$.

5 The case $\ell = 2$.

Now we consider the situation $\ell = 2$. In this case, for $c + c' - 2 > 2d$ and $t \geq 3$ a complete information on the integer m is not yet known. In this section we find the integer m in function of the possible values of the parameter t. As a consequence we deduce also the value of m for semigroups with Cohen-Macaulay type $\tau = 3$ and for semigroups with $e \leq 6$. First we prove and recall some facts.

Lemma 5.1 *Let τ be the Cohen-Macaulay type of S. Then:*

(1) *If $c - e \leq c' \leq c - e + 1$, then S is acute.*

(2) *Every gap $h \geq c - e$ belongs to $S(1) \setminus S$, in particular* $\{d+1, ..., d+\ell\} = \{c-\ell, ..., c-1\} \subseteq S(1) \setminus S$.

(3) $\tau \geq \ell$ *and if* $\tau = \ell$, *then* $c' = c - e$.

(4) *If $t > 0$, for each $k \in \mathbb{N}$, $0 \leq k \leq t-1$ such that $d - k \in S$, we have $e \neq 2\ell + k$. Further*

(a) *If $t \geq 2$ and $c' < d$, then $e \neq 2\ell,\ e \neq 2\ell + 1$,*

(b) *If $c' \leq d - t \leq d$, then $e \neq 2\ell + k$, for each $k \in \{0, ..., t-1\}$.*

Proof. Items (1)-(3) are proved in [4, 4.10 and 4.11].
(4). Let k be as above. Then: $c-2\ell-k-1=(c-\ell-1)-k-\ell=d-k-\ell\in S$ by (2.7.4). If $e=2\ell+k$, then $c-1-e\in S$, hence also $c-1\in S$, impossible.
(a) follows by applying (4): if $c'<d$, one has $\{d-1,d\}\subset S$. Also (b) is immediate by (4). $\diamond$

Proposition 5.2 *Assume* $c+c'-2>2d$ *and* S *non-acute. Then:*

(1) $d-c'+2\leq\ell\leq e-3-(d-c')$.

(2) $e\geq 5+2(d-c')$, *and if* $e\in\{5,6\}$, *then* $c'=d$.

Proof. (1). The first inequality is proved in (2.7.2). In order to obtain the second one note that $c'\geq c-e+2$ (by (5.1.1), since S is non-acute and by (2.7.1)). Then recall the equality $c=d+\ell+1$.
(2) is immediate by (1). $\diamond$

Lemma 5.3 *Assume* $\ell=2$. *Then:*

(1) $c+c'-2>2d \iff d=c'$.

(2) *The following conditions*

  (a) $c+c'-2>2d$ *and* $t>0$,

  (b) $d'=d-2$.

  (c) $d=c'$ *and* $t\geq 2$,

  are equivalent and imply: $t\geq 3\iff d-4\in S$.

Proof. (1) follows immediately from (2.7.2) in the case $\ell=2$.
(2). $(a)\Longrightarrow(b)$. If (a) holds, we have $d-1\notin S$ (since $c'=d$ by (1)) and $d-2=d-\ell\in S$ (because $t>0$); then $d'=d-2$.
$(b)\Longrightarrow(c)$. The assumption $d'=d-2$ implies:
  $d-1\notin S$, so that $c'=d$;
  $d-\ell=d-2\in S$. Then $t\geq 2$.
$(c)\Longrightarrow(a)$ follows by (1).
Further, if the conditions of (2) hold, then: $t\geq 2$ and $t=2\iff d-4=d-\ell-2\notin S$ (because $d-\ell=d-2\in S$ and $d-1\notin S$ since $d=c'$). $\diamond$

Lemma 5.4 *Suppose* $\ell=2$, $t\geq 3$, $\{d-3,d-2\}\subset S$. *Then one has:* $[d-t-1,d-2]\cap\mathbb{N}\subset S$.

Proof. By the assumption $\{d-3,d-2\}\subset S$, it is enough to show that $d-k-1\in S$ for $3\leq k\leq t$: we prove by induction that if $[d-k,d-2]\cap\mathbb{N}\subset S$, then $d-k-1\in S$. Since $d-(k-1)\in S$ and $k-1<t$, then $d-k-1=d-2-(k-1)=d-\ell-(k-1)\in S$, by (2.7.4). $\diamond$

Theorem 5.5 *Suppose* $\ell=2$. *Then the parameter* m *takes the following values.*

$$(1)\ \ s_m = 2d - t \quad if \left[\begin{array}{ll} c + c' - 2 \leq 2d, \quad or & \\ c + c' - 2 > 2d \quad and & \left[\begin{array}{ll} or & t \leq 2, \\ or & t = 4 \\ or & t \geq 5 \quad and \quad d - 3 \in S. \end{array}\right. \end{array}\right.$$

$$(2)\ \ s_m = 2d - 4 \quad if \ \ c + c' - 2 > 2d \quad and \left[\begin{array}{lll} either & t = 3 \quad and & d - 6 \notin S \\ or & t \geq 5 \quad and & d - 3 \notin S. \end{array}\right.$$

(3) $s_m = 2d - 6$ *if* $c + c' - 2 > 2d$, $t = 3$ *and* $d - 6 \in S$
(*all the remaining cases*).

Proof. For the cases $c + c' - 2 \leq 2d$ and $c + c' - 2 > 2d$ with $t \leq 2$ see (2.8.1). Hence from now on assume that $c + c' - 2 > 2d$ and $t \geq 3$: again by (2.8) we know that $s_m \leq 2d - 3$. By (5.3.2) S verifies:

$$d - 1 \notin S, \quad d - 2 \in S, \quad d - 4 \in S, \quad c' = d.$$

Further for $s_i \leq 2d - 2$ we have $B(s_i) = B(s_{i+1}) = \emptyset$ and $\beta(s_i) = 0$ (by (3.4), since $2c' - 2 = 2d - 2$ and $d' > 0$).

Case $t = 3$. Since $\{d - 1, d - 2\} \cap S = \{d - 2\}$, we have $m \leq 2d - 4 - g$ (2.8.2). Further one has: $d - 5 = d - \ell - t \notin S$.
Consider $s = 2d - 4$; then $s + 1 - c = d - 6$ and by (5.3) we have

$$s = 2d', \quad s - d = d - 4 \in S, \quad s + 1 - c' = s + 1 - d = d - 3 = d - t \in S.$$

If $d - 6 \notin S$ one has $\nu(s+1) < \nu(s)$ by the 2-nd row in table 3.11.(1), $s_m = 2d - 4$. If $d - 6 \in S$ one has $\nu(s + 1) > \nu(s)$ by the 6-th line in the same table. In this second case consider $s \leq 2d - 5$: we have

$$C(2d - 6) = \{(d - 2, d - 4), (d - 3, d - 3), (d - 4, d - 2)\}$$
$$C(2d - 5) = \{(d - 2, d - 3), (d - 3, d - 2)\}$$
$$C(2d - 4) = \{(d - 2, d - 2)\}.$$

Note also:

$$2d - 6 - d = d - 6 \in S,\ 2d - 5 - d = d - 5 \notin S,\ 2d - 4 - c' = 2d - 4 - d = d - 4 \in S.$$

Then (table 3.10.(a)): $\nu(2d - 4) - \nu(2d - 5) \geq \gamma(2d - 5) + 2 = 1$. Moreover $\alpha(2d - 6) = -2$, $\gamma(2d - 6) = -1$; since $\beta(2d - 6) = 0$, $\delta(2d - 6) \leq 2$ then we have $\nu(2d - 5) - \nu(2d - 6) < 0$, and so $s_m = 2d - 6$.

The *case* $t = 4$ and the *case* $t \geq 5$, $d - 3 \notin S$ follow by (2),(3) of Th. 2.8. Finally we have to consider

the *case* $t \geq 5$, $d - 3 \in S$: we already know that $m \leq 2d - 5 - g$ by (2.8.3). For $s = 2d - k$, $5 \leq k \leq t$ we have $s \geq c$, since $s \geq d + (d - t) \geq c$ by (2.7.5). Further: $d - t \leq d - k = s - d \leq d - 5$; it follows that
$d - t + 1 \leq d - k + 1 = s + 1 - d = s + 1 - c' \leq d - 4$. Therefore:

$s - d \in S'$, $s + 1 - c' \in S'$, $s + 1 - c = d - \ell - k \in S$, if $k \neq t$,
$s + 1 - c \notin S$ if $k = t$ (by Lemma 5.4 and by (2.7.4)).

Now by table 3.10.(a): $\nu(s + 1) - \nu(s) = \left[\begin{array}{ll} \gamma(s) & if\ k = t \\ \gamma(s) + 2 & if\ k \neq t \end{array}\right.$.

The required result follows if we prove the
Claim. $\gamma(s) = -1$ *for each* $s = 2d - k$, $5 \leq k \leq t$.
Proof of the claim. If $t = 5$ and $d - 3 \in S$, then $\gamma(2d - 5) = -1$. In fact

$$C(2d - 5) = \{(d - 2, d - 3), (d - 3, d - 2)\}$$

$C(2d-4)=\{(d-2,d-2)\}$.
If $t \geq 6$ and $d-3 \in S$, since $[d-t-1, d-2] \cap \mathbb{N} \subset S$ by (5.4) and $d-1 \notin S$, for each $k \in \mathbb{N}$, $5 \leq k \leq t$ one has:
$C(2d-k)=\{(d-2,d-k+2),(d-3,d-k+3),...,(d-k+2,d-2)\}$,
$C(2d-k+1)=\{(d-2,d-k+3),...,(d-k+3,d-2)\}$.
Hence $\#C(2d-k+1)=\#C(2d-k)-1$, so that $\gamma(2d-k)=-1$. $\diamond$

As a first consequence we know the integer m for every semigroup S of multiplicity $e \leq 6$:

Corollary 5.6 (1) *If* $e \leq 4$, *then* $s_m = 2d-t$.

(2) *If* $e \in \{5,6\}$, *then either* $\ell = 2$ *and Theorem* 5.5 *holds, or* $m = 2d-t-g$.

Proof. If $e \leq 4$, we cannot have $c+c'-2 > 2d$ and $t > 0$, by (5.2.1) and (2.3.4). Hence either $c+c'-2 \leq 2d$, or $c+c'-2 > 2d$ and $t = 0$: the claim follows by (2.8.1).
If $e = 5, 6$, we can assume that $c+c'-2 > 2d$ and $t \geq 3$, since in the other cases the result follows by (2.8.1). Then, if $e = 5$, by (5.2) one gets $\ell = 2$;
If $e = 6$, again by (5.2) it follows that $\ell \in \{2,3\}$. The case $\ell = 3$, with $t \geq 3$ is impossible by (5.1.4), (applied with $k = 0$). $\diamond$

5.1 The case τ=3.

For a semigroup having Cohen-Macaulay type $\tau = 3$, Theorem 5.5 allows to complete the partial results of [4, Prop. 4.16]. This is shown in the following proposition.

Proposition 5.7 *Suppose that* S *has Cohen-Macaulay type* $\tau = 3$. *Then* $e \geq 4$ *and*

(1) *Either S is acute, or* ($\ell = 2$, $d' = c'-2$ *and* $S(1)\setminus S = \{c'-1, d+1, d+2\}$).

(2) *When* $\ell = 2$, $d' = c'-2$ *and* $c+c'-2 > 2d$, *we have:*

- (*a*) $S(1) \setminus S = \{d-1, d+1, d+2\}$.
- (*b*) $t \geq max\{2, e-4\}$.
- (*c*) *If* $t > e-4$, *then* $e = 5$.

In particular, if $t \geq 3$, *then:*

- (*d*) $t = e-4$ *and* $d-t-3 \notin S$.
- (*e*) $d-3 \in S$.

Proof. (1). If $\tau = 3$ obviously $e \geq 4$, by the well-known inequality $\tau \leq e-1$. Further, by (5.1) we get $\ell \leq 3$ and either S is acute, or $\ell = 2$, $d' \geq c'-2 \geq c-e$ (otherwise S is acute) that means $d' = c'-2$, $c'-1 \in S(1) \setminus S$ (since $c'-1 \notin S$). Hence (1) is true.

(2). First note that $c' = d$ by (5.3.1). This implies $t \geq 2$ by (5.3.2).
Further $d-1 \in S(1)$, since for each $s \in S \setminus \{0\}$, one has

$d-1+s \geq d-1+e \geq d+3 = c$, then $d-1+s \in S$.

Moreover $\{d+1, d+2\} \subseteq S(1) \setminus S$ by (5.1.2). Since $\#(S(1) \setminus S) = \tau = 3$, one has $S(1) \setminus S = \{d-1, d+1, d+2\}$ hence (a) holds.
(b) If $t \leq e-5$, then $d-\ell-t \geq c-e$; in fact $d-\ell-t = d-2-t$ and we get $d-\ell-t \geq d-2-(e-5) = d+3-e = c-e$. This fact implies: $d-\ell-t \in S(1) \setminus S$ by (5.1.2). On the other hand, by the assumption $\ell = 2$, we have: $d-\ell-t \leq d-2 < d-1$, that contradicts (a). It follows $t \geq e-4$.
(c) If $t > e-4$, by (2.2) one has:

$$\left[\begin{array}{lll} \textit{either} & (i) & d-(e-4) \notin S, \\ \quad \textit{or} & (ii) & d-(e-4) \in S \text{ and } d-\ell-(e-4) \in S. \end{array}\right.$$

When (i) holds, since $d-(e-4) = c+1-e$, we get $d-(e-4) \in S(1) \setminus S$ (see (5.1.2)). Then, by (a), since $d-(e-4) < d$ one necessarily has $d+4-e = d-1$, i.e., $e = 5$.
It is easily seen that (ii) cannot happen. In fact, $d-\ell-(e-4) = d+2-e = c-1-e \notin S$.
(d). Under the assumptions of (2), let $t \geq 3$. Then we have also $d-4 \in S$ by (5.3). It follows that $t = e-4$. In fact by (b) above, $t \geq e-4$; but $t > e-4$ implies $e = 5$ (see (c)), hence $d+1 = d-4+e \in S$, absurd. In particular, since $t = e-4$, one has $d-t-3 \notin S$, otherwise $d+1 = d-t-3+e \in S$, a contradiction. This proves (d).
To see (e), note that by (d), $e-4 = t \geq 3 \Longrightarrow e \geq 7$. Hence $d-3 \in S(1)$; in fact for every $s \in S \setminus \{0\}$ one has $d-3+s \geq d-3+7 = d+4 > c$, i.e., $d-3+s \in S$. Using (d) and $(2a)$ we deduce that $d-3 \in S$. $\diamond$

Example 5.8 We show a semigroup which satisfies the conditions of (5.7.2):
$S = \{0, 7_e, 10, 14, 15_{d'}, 17_{c'=d}, 20_c \rightarrow\}$.
Then: $\ell = 2$, $d = c'$, $d-\ell = 15 \in S$, $d' = d-2 = d-t$, $d'-\ell \notin S$, $t = 2$.
Further, $S(1) \setminus S = \{16, 18, 19\}$ and so $\tau = 3$, $c+c'-2 = 35 > 2d = 34$.

Corollary 5.9 *Assume that S has Cohen-Macaulay type $\tau = 3$. Then:*

$$\left[\begin{array}{ll} s_m = 2d-4, & \textit{if } t = 3 \textit{ and } S \textit{ is non-acute}, \\ s_m = 2d-t & \textit{in all the remaining cases.} \end{array}\right.$$

Proof. If S is acute, by (2.3.4) we can use use (2.8.1). For the remaining cases, if $c+c'-2 \leq 2d$, see (2.8.). If $c+c'-2 > 2d$, by (5.7) we have

$$\ell = 2, \quad t \geq 2 \quad \text{and} \quad \left[\begin{array}{ll} \textit{or} & t = 2, \\ \textit{or} & t = 3 \text{ and } d-6 \notin S, \\ \textit{or} & t = 4, \\ \textit{or} & t \geq 5 \text{ and } d-3 \in S, \end{array}\right.$$

then it suffices to apply Theorem 5.5. $\diamond$

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