On some invariants in numerical semigroups and estimations of the order bound.

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[1] **Abstract.** Let $S = \{s_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ be a numerical semigroup. For $s_i \in S$, let $\nu(s_i)$ denote the number of pairs $(s_i - s_j, s_j) \in S^2$. When S is the Weierstrass semigroup of a family $\{\mathcal{C}_i\}_{i\in\mathbb{N}}$ of one-point algebraic-geometric codes, a good bound for the minimum distance of the code $\mathcal{C}_i$ is the Feng and Rao *order bound* $d_{ORD}(C_i)$. It is well-known that there exists an integer m such that $d_{ORD}(C_i) = \nu(s_{i+1})$ for each $i \geq m$. By way of some suitable parameters related to the semigroup S, we find upper bounds for m and we evaluate m exactly in many cases. Further we conjecture a lower bound for m and we prove it in several classes of semigroups.

Index Therms. Numerical semigroup, Weierstrass semigroup, AG code, order bound on the minimum distance, Cohen-Macaulay type.

1 Introduction.

Let $S = \{s_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ be a numerical semigroup and let $e,\ c,\ c',\ d,\ d'$ denote respectively the multiplicity, the conductor, the subconductor, the dominant of the semigroup and the greatest element in S preceding c' (if $e > 1$), as in Setting 2.1. Further let ℓ be the number of gaps of S between d and c, and let

$$\widetilde{s} := max\{s \in S \ \ such\ that\ \ s \leq d \ \ and\ \ s - \ell \notin S\}.$$

When S is the Weierstrass semigroup of a family $\{\mathcal{C}_i\}_{i\in\mathbb{N}}$ of one-point AG codes (see [3],[2]), a good bound for the minimum distance of $\mathcal{C}_i$ is the Feng and Rao *order bound*

$$d_{ORD}(C_i) := min\{\nu(s_j) : \ j \geq i + 1\}$$

where, for $s_j \in S$, $\nu(s_j)$ denotes the number of pairs $(s_j - s_k, s_k) \in S^2$. It is well-known that there exists an integer m such that sequence $\{\nu(s_i)\}_{i\in\mathbb{N}}$ is non-decreasing for $i \geq m + 1$ (see[7]) and so

$$d_{ORD}(C_i) = \nu(s_{i+1}) \text{ for } \ i \geq m.$$

For this reason it is important to find the element s_m of S. In our papers [5] and [6], we proved that $s_m = \widetilde{s} + d$ if $\widetilde{s} \geq d'$, and we evaluated s_m in cases $\ell \leq 2$, or $e \leq 6$, or *Cohen-Macaulay type* ≤ 3.

In this paper, by a more detailed study of the semigroup we find interesting relations among the integers defined above; further by using these relations we deduce the Feng and Rao order bound in several new situations. Moreover in every considered case we show that $s_m \geq c + d - e$.

In Section 2, we establish various formulas and inequalities among the integers $e,\ \ell,\ d',\ c',\ d,\ c$ and $t := d - \widetilde{s}$, see in particular (2.5) and (2.6).

In Section 3, by using the results of Section 2 and some result from [6], we improve the known facts on s_m recalled above; further we state for each semigroup and we prove in many cases the

conjecture: $\quad s_m \geq c + d - e.$

In Section 4 we treat some particular cases: for each of them we also prove that the conjecture holds.

In conclusion we see that the value of the *order bound* s_m depends essentially on the position of the

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integer $\widetilde{s}$ in the semigroup. We summarize below the main results for the convenience of the reader.

$$\begin{array}{lll} \text{If} \quad \widetilde{s} \geq d' + c' - d & \text{then} & s_m = \widetilde{s} + d \\ \text{if} \quad \widetilde{s} = 2d' - d & \text{then} & s_m = \widetilde{s} + d = 2d' \\ \text{if} \quad \widetilde{s} \leq d' + c' - d - 1 & \text{then} & s_m \leq max\{\widetilde{s} + d, 2d'\} \qquad (3.4). \end{array}$$

When $\widetilde{s} \leq d' + c' - d - 1$, we prove the following partial answers.

$$\text{If} \quad [d' - \ell, d'] \cap \mathbb{N} \subseteq S \qquad \text{then} \quad \left[\begin{array}{ll} \widetilde{s} + d' - \ell + 1 \leq s_m \leq 2d' & if \;\; 2d' - d < \widetilde{s} < d' + c' - d \\ s_m = \widetilde{s} + d & otherwise. \qquad (3.11) \end{array} \right.$$

$$\text{If} \quad \left\{ \begin{array}{l} \widetilde{s} \leq d' - 2 \\ [\ \widetilde{s} + 2, d'\ ] \cap \mathbb{N} \subseteq S \\ 2d' - d < \widetilde{s} < d' + c' - d \end{array} \right. \qquad \text{then} \quad \left[\begin{array}{ll} s_m = \widetilde{s} + d & \iff \widetilde{s} + 1 \notin S \text{ and } c' = d \\ s_m \leq \widetilde{s} + d - 1 & otherwise. \qquad (3.8) \end{array} \right.$$

$$\text{If} \quad \left\{ \begin{array}{l} \widetilde{s} \leq 2d' - d \\ [\ \widetilde{s} + 2, d'\ ] \cap \mathbb{N} \subseteq S \end{array} \right. \qquad \text{then} \quad s_m = \widetilde{s} + d. \qquad (3.8)$$

Finally we consider several particular subcases: if H denotes the subset of gaps of S inside the interval $[c - e, c' - 1]$ and τ is the Cohen-Macaulay type of S, we deduce the exact value or good estimations for s_m in the following situations.

$$\begin{array}{lll} \text{If} \quad H = \emptyset, & \text{then} & s_m = \widetilde{s} + d \quad (4.1) \\ \text{If} \quad H \; is \; a \; non \; empty \; interval, & \text{then} & s_m = \left[\begin{array}{ll} 2d' & if \;\; \widetilde{s} \geq 2d' + 1 - d \\ \widetilde{s} + d & otherwise \quad (4.1) \end{array} \right. \\ \text{If} \quad S \; is \; associated \; to \; a \;\; Suzuki \; curve, & \text{then} & s_m = \widetilde{s} + d \quad (4.14). \end{array}$$

If $\#H \leq 2$, see (4.4).

If $\ell \leq 3$, see (4.5), (4.7).

If $\tau \leq 7$, see (4.10).

If $e \leq 8$, see (4.11).

If *S is generated by an Almost Arithmetic Sequence and* $embdim(S) \leq 5$, see (4.12).

2 Semigroups: invariants and relations.

We begin by giving the setting of the paper.

Setting 2.1 In all the article we shall use the following notation. Let $\mathbb{N}$ denote the set of all non-negative integers. A *numerical semigroup* is a subset S of $\mathbb{N}$ containing 0, closed under summation and with finite complement in $\mathbb{N}$; we shall always assume $S \neq \mathbb{N}$. We denote the elements of S by $\{s_i\}$, $i \in \mathbb{N}$, with $s_0 = 0 < s_1 < ... < s_i < s_{i+1} ....$

We list below some invariants and subsets related to a semigroup $S \subset \mathbb{N}$ we shall need in the sequel.

$$
\begin{array}{rcl}
e & := & s_1 > 1, \text{ the } \textit{multiplicity} \text{ of } S. \\
c & := & min\ \{r \in S \mid r + \mathbb{N} \subseteq S\}, \text{ the } \textit{conductor} \text{ of } S \\
d & := & \text{the greatest element in } S \text{ preceding } c, \textit{ the dominant} \text{ of } S \\
c' & := & max\{s_i \in S \mid s_i \leq d \ \ and \ \ s_i - 1 \notin S\}, \text{ the } \textit{subconductor} \text{ of } S \\
d' & := & \text{the greatest element in S preceding } c', \text{ when } \ c' > 0 \\
\ell & := & c - 1 - d, \text{ the number of gaps of } S \text{ greater than } d \\
g & := & \#(\mathbb{N} \setminus S), \text{ the } \textit{genus} \text{ of } S \ (= \text{the number of gaps of } S) \\
S' & := & \{s \in S \mid s \leq d'\} \subseteq S \\
S(1) & := & \{b \in \mathbb{N} \mid \ b + (S \setminus \{0\}) \subseteq S\} \\
\tau & := & \#(S(1) \setminus S), \text{ the } \textit{Cohen–Macaulay type} \text{ of } S \\
H & := & [c - e, c'] \cap \mathbb{N} \setminus S \quad \subseteq \mathbb{N} \setminus S.
\end{array}
$$

(Note that $c - e \leq c'$ since $c - e - 1 \notin S$).
According to this notation we can represent a semigroup S with $c' > 0$ as follows:

$$S = \{0, * \ldots *,\ e,\ \ldots,\ d', * \ldots *,\ c' \ \longleftrightarrow \ \overset{\ell \text{ gaps}}{d, * \ldots *},\ c \rightarrow\} = S' \cup \{\ c' \ \longleftrightarrow \ d, * \ldots *,\ c \rightarrow\},$$

where " $*$ " indicates *gaps*, " $* \ \ldots \ *$ " *interval of all gaps*, and " $\longleftrightarrow$ " *interval without gaps.*

Moreover for $s_i \in S$ we fix the following notation.

$$
\begin{array}{lcll}
N(s_i) & := & \{(s_j, s_k) \in S^2 \mid s_i = s_j + s_k\}; & \nu(s_i) \ := \ \#N(s_i); \\
\eta(s_i) & := & \nu(s_{i+i}) - \nu(s_i). & \\
d_{ORD}(i) & := & min\{\nu(s_j) \mid j > i\}, \text{ the } \textit{order bound.} & \\
A(s_i) & := & \{(x, y), (y, x) \in N(s_i) \mid x \leq d',\ c' \leq y \leq d\}; & \alpha(s_i) \ := \ \#A(s_{i+1}) - \#A(s_i). \\
B(s_i) & := & \{(x, y) \in [c', d]^2 \cap N(s_i)\}; & \beta(s_i) \ := \ \#B(s_{i+1}) - \#B(s_i). \\
C(s_i) & := & \{(x, y) \in S'^2 \cap N(s_i)\}; & \gamma(s_i) \ := \ \#C(s_{i+1}) - \#C(s_i). \\
D(s_i) & := & \{(x, y),\ (y, x) \in N(s_i) \mid \ x \geq c,\ x \geq y\}; & \delta(s_i) \ := \ \#D(s_{i+1}) - \#D(s_i).
\end{array}
$$

Now we recall some definition and former results for completeness. First, a semigroup S is called

ordinary *if* $S = \{0\} \cup \{n \in \mathbb{N},\ n \geq c\}$

acute *if* *either* S *is ordinary, or* c, d, c', d' *satisfy* $c - d \leq c' - d'$ [1, Def. 5.6].

Definition 2.2 *We define the invariants* $\widetilde{s}$, m *and* t *as follows.*

$\widetilde{s} \ := \ max\ \{s \in S \ such\ that \ \ s \leq d \ \ and\ s - \ell \notin S\}.$

$t \ := \ d - \widetilde{s}.$

$m := \ min\ \{j \in \mathbb{N} \ such\ that\ the\ sequence\ \{\nu(s_i)\}_{i \in \mathbb{N}}\ is\ non\text{-}decreasing\ for\ i > j\}$

$(m > 0 \iff \nu(s_m) > \nu(s_{m+1}) \ and\ \nu(s_{m+k}) \leq \nu(s_{m+k+1}),\ \ for\ each\ k \geq 1).$

Theorem 2.3 *Let* $S = \{s_i\}_{i \in \mathbb{N}}$ *be as in Setting 2.1.*

(1) $\nu(s_i) = i + 1 - g$, *for every* $s_i \geq 2c - 1$. [7, Th. 3.8]

(2) $\nu(s_{i+1}) \geq \nu(s_i)$, *for every* $s_i \geq 2d + 1$. [5, Prop. 3.9.1]

(3) *If* S *is an ordinary semigroup, then* $m = 0$. [1, Th. 7.3]

(4) *If* $\widetilde{s} \geq d'$, *then* $s_m = \widetilde{s} + d$ [6, Th. 4.1, Th.4.2].

In particular:

(a) *if* $t \leq 2$, *then* $s_m = \widetilde{s} + d$,

(b) *if* S *is an acute semigroup, then* $s_m = \widetilde{s} + d$, *with*

(i) *either* $d - c' \geq \ell - 1, \;\; s_m = c + c' - 2 = \widetilde{s} + d,$
(ii) *or* $\widetilde{s} = d \;\; (s_m = 2d).$ [5, Prop. 3.4] .

(5) *If* $c' \in \{c - e, c - e + 1\}$, *then* S *is acute.* [6, Lemma 5.1].

Remark 2.4 (1) By the definition of $\widetilde{s}$ it is clear that:

$$s - \ell \in S \;\; \textit{for each} \;\; s \in S \;\; \textit{such that} \;\; \widetilde{s} < s \leq d.$$

(2) Theorem 2.3 implies that $0 < s_m \leq 2d$ for every non-ordinary semigroup.
(3) The condition (a) of (2.3.4) does not imply S acute; analogously there exist non-acute semigroups satisfying the conditions $(4.b, i - ii)$, see Example 2.9.2.

We complete this section with some general relation among the invariants defined above.

Proposition 2.5 [6, Prop. 2.5] *Let* $c' = c - e + q$, *with* $q \geq 0$. *Then*

(1) $e \leq 2\ell + t + q.$

(2) *The following conditions*

- (a) $d - c' \geq \ell - 1 \quad (\textit{ i.e. } \;\; c + c' - 2 \leq 2d).$
- (b) $\widetilde{s} - \ell = c' - 1.$
- (c) $c + c' - 2 = \widetilde{s} + d.$
- (d) $e = 2\ell + t + q$

are equivalent and imply

- (i) $c' \leq \widetilde{s} \leq d \qquad (\Longrightarrow s_m = \widetilde{s} + d).$
- (ii) S *is acute* $\iff d - d' \geq 2\ell + t.$

Proof. (1) By definition 2.2 we have $\widetilde{s} - \ell \leq c' - 1 = c - e + q - 1$, then $\widetilde{s} - \ell \leq d + \ell - e + q$ and so $e \leq 2\ell + t + q$.
(2) The equivalences $(2.a) \iff (2.b) \iff (2.c)$ are proved in [6, Prop. 2.5]. Clearly the equality $e = 2\ell + t + q$ holds if and only if $\widetilde{s} - \ell = d - t - \ell = c' - 1$. Further:
(i) is obvious by $(2.b)$.
(ii) If $(2.b)$ holds, then $d - d' - (2\ell + t) = \widetilde{s} - \ell - d' - \ell = (c' - d') - (\ell + 1) = (c' - d') - (c - d)$. Then S is acute $\iff d - d' \geq 2\ell + t$.

Theorem 2.6 *Let* $t = d - \widetilde{s}$ *(see* 2.1*). The following facts hold.*

(1) (a) *If* $0 \leq h < e$ *and* $d - h \in S$, *then* $e \geq h + \ell + 1.$
  (b) *If* $s, s' \in S, \;\; s \geq c - e$ *and* $s - \ell \leq s' < s$, *then* $s' \geq c - e.$

(2) $\widetilde{s} \geq c - e \quad (\textit{equivalently}, \;\; e \geq t + \ell + 1).$

(3) *Let* $t > 0$ *and let* $s' := min\{s \in S \mid s > \widetilde{s}\}$. *Then*

$e \geq 2\ell + 1 + d - s' \geq 2\ell + 1 \quad (\textit{equivalently, } s' \geq c - e + \ell).$

In particular,

- (a) $\widetilde{s} + 1 \in S \;\; \Longrightarrow \;\; e \geq 2\ell + t;$
- (b) $c' \leq \widetilde{s} < d \Longrightarrow e \geq 2\ell + t.$

(4) *One of the following conditions hold*

- (a) $\widetilde{s} - \ell \geq c - e \;\; (\textit{equivalently} \;\; e > 2\ell + t, \;\textit{equivalently} \;\; \widetilde{s} - \ell \in H)$
- (b) $\widetilde{s} - \ell = c - e - 1 \;\; (\textit{equivalently} \;\; e = 2\ell + t)$
- (c) $c - e - \ell \leq \widetilde{s} - \ell < c - e - 1 \;\; (\textit{equivalently} \;\; e < 2\ell + t)$

(5) *Assume* $e < 2\ell + t$, *then* :

(*a*) *either* $\widetilde{s} \le d'$ *or* $t = 0$;

(*b*) *in case* $\widetilde{s} \le d'$ *we have*: $[\widetilde{s}+1, c-e+\ell-1] \cap S = \emptyset$, $\#H \ge 2\ell + t - e$, *further if* $\widetilde{s} < d'$, *then* $\#H \ge 2\ell + t - e + 1 \ge 2$.

Proof. (1.*a*) We have $d < d - h + e \in S$. Hence $d - h + e \ge c = d + \ell + 1$.
(1.*b*) If $s \ge c$ we have $s' \ge c$. If $s \le d$, let $s' = d - h$, $s = d - k \ge c - e$ (hence $k + \ell \le e - 1$), then $d - \ell - k \le d - h \Longrightarrow h \le k + \ell \le e - 1$. Now apply (*a*): $e \ge h + \ell + 1$, equivalently, $d - h \ge d + \ell + 1 - e$.
(2) Let $d = c - e + h\ell + r$, with $h \ge 0$, $0 \le r < \ell$ (recall that $c - e \le d$). If $\widetilde{s} < c - e$, by (2.4.1) we get $\widetilde{s} < d - h\ell \in S$, $d - (h+1)\ell \in S$; further we get $c - e - \ell \le d - (h+1)\ell < c - e$, a contradiction because $[c - \ell - e, c - e - 1] \cap S = \emptyset$ for every semigroup.
(3) By (2.4.1), $\widetilde{s} < s' \le d \Longrightarrow s' - \ell \in S$ and so $s' - \ell + e \in S$. Since $c - e \le \widetilde{s} < s'$, we get $s' - \ell + e > c - \ell = d + 1$; it follows that $s' - \ell + e \ge c = d + \ell + 1$.
(4) Since $c - e - 1 \notin S$, the statements are almost immediate by (2).
(5.*a*) follows by ((3.*b*).
(5.*b*) We are in case (4.*c*): since obviously $[c - e - \ell, c - e - 1] \cap S = \emptyset$, we have

$$[\widetilde{s} - \ell + 1, c - e - 1] \cap \mathbb{N} \subseteq [c - \ell - e + 1, c - e - 1] \cap \mathbb{N} \subseteq \mathbb{N} \setminus S.$$

By the definition of $\widetilde{s}$, we deduce that $[\widetilde{s} + 1, c - e + \ell - 1] \cap \mathbb{N} \subseteq H$. The last inequality follows recalling that $\widetilde{s} + 1 < d' \Longrightarrow c - e + \ell - 1 < d'$, hence $d' + 1 \in H \setminus [\widetilde{s} + 1, c - e + \ell - 1]$. ⋄

Corollary 2.7 *Assume* $\widetilde{s} < d$ (*i.e.* $t > 0$). *Then*

(1) *If* $c' = c - e + q$, *with* $q \in \{0, 1\}$, *then* $d - c' \ge \ell - 1$ *and* $e = 2\ell + t + q$.

(2) *If* $d - c' \le \ell - 2$, *then*

(*a*) $d - c' + 2 \le d - d' \le \ell \le e - 3 - (d - c')$;

(*b*) *if* $\widetilde{s} \ge 2d' - d$, *then* $t \le 2\ell$.

Proof. (1) By the assumptions and by (5), (4) of Theorem 2.3, we have $d - c' \ge \ell - 1$. Then the other statement follows by (2.5.2).
(2.*a*) Since $d' \le c' - 2$ the first inequality holds for any semigroup. We have $d - c' \le \ell - 2$, by assumption, and $d - \ell \in S$, by (2.4.1). Hence $d' \ge d - \ell$. For the last inequality see [6, Prop. 5.2]].
(2.*b*) follows by (2.*a*) because the assumption means $t \le 2d - 2d'$. ⋄

Corollary 2.8 *Assume* $\widetilde{s} \le d'$. *The following facts hold:*

(1) $d - c' \le \ell - 2$, $d - d' \le \ell$, $c' \ge c - e + 2$.

(2) *If* $e \le 2\ell + t$, *then*

(*a*) $\#H \le \ell + t - 2(d - c') - 4$.

(*b*) $e = 2\ell + t \Longleftrightarrow d + 2\ell - e \in S$.

(3) *If* $H \subseteq [d' - t + 1, c' - 1]$, *then* $e \le 2\ell + t$.

(4) *If* $H = [d' + 1, c' - 1] \cap \mathbb{N}$ *and* $e < 2\ell + t$, *then* $\widetilde{s} = d'$.

Proof. (1) By (2.5.2) we see that $\widetilde{s} \le d' \Longrightarrow d - c' \le \ell - 2$, therefore $d - d' \le \ell$ (2.7). Further we have $c' \ge c - e + 2$ because $c - e \le \widetilde{s}$ (2.6.2) and $\widetilde{s} \le d' \le c' - 2$.
(2*a*) By (1) we have $d - d' \le \ell$, then by (2.6.1-2) and by (2.4.1), we deduce that

$$\{c' - \ell, ..., d - \ell, \widetilde{s}, d'\} \subset S \cap [c - e, d'].$$

Hence $\#H \le c' - (c - e) - 2 - (d - c' + 1) = 2c' - 2d - \ell - 1 - 3 + e \le 2c' - 2d - \ell - 1 - 3 + 2\ell + t = \ell + t - 2(d - c') - 4$.
(2*b*) Clearly $e = 2\ell + t \Longrightarrow d + 2\ell - e = \widetilde{s} \in S$. The converse follows by the assumption and by Theorem 2.6.5*b*: $c - e + \ell - 1 = d + 2\ell - e \in S \Longrightarrow e \ge 2\ell + t$; then $e = 2\ell + t$.
(3) $\widetilde{s} \le d' \Longrightarrow d - \ell \le d'$ by (1). Hence $\widetilde{s} - \ell = d - \ell - t \le d' - t$: now the assumption on H implies $\widetilde{s} - \ell \notin H$, i.e., $e \le 2\ell + t$ (2.6.4).
(4) If $e < 2\ell + t$, we have $\widetilde{s} + 1 \notin S$ (2.6.3); since $c - e < \widetilde{s} + 1$ we get $\widetilde{s} + 1 \in H$, and so $\widetilde{s} = d'$. ⋄

Example 2.9 (1) If $t > 0$, for each s_i such that $\widetilde{s} < s_i \le d$, we have that $s_i - \ell \in S$, hence $s_i - s_{i-1} \le \ell$, but it is not true that for each $s_i \in S$ such that $c - e \le s_i < d$, we have $s_{i+1} - s_i \le \ell$: for instance let $\ell \ge 2$ and $S = \{0, 5\ell_e,\ 7\ell_{\widetilde{s}=d-\ell=d'},\ 8\ell_d,\ 9\ell+1_c \rightarrow\}$.

(2) When $t = 0$ the inequality $e \ge 2\ell + 1$ (proved in (2.6.3) for $t > 0$) in general is not true, even for acute semigroups:

$S_1 = \{0,\ 10_{e=d'},\ 17_{c'}, 18,\ 19,\ 20_d,\ 27_c \rightarrow\}$:

$\ell = 6,\ t = 0,\ \widetilde{s} = d$ and S_1 is acute with $d - c' \le \ell - 2,\ e < 2\ell$.

$S_2 = \{0,\ 8_e,\ 12_{d'},\ 14_{c'},\ 15,\ 16_d,\ 20_c \rightarrow\}$:

$\ell = 3,\ t = 0$, S_2 is non-acute with $d - c' = \ell - 1,\quad e > 2\ell$.

$S_3 = \{0,\ 7_e,\ 12_{d'},\ 14_{c'=d},\ 19_c \rightarrow\}$:

$\ell = 4,\ t = 0,\quad S_3$ is non-acute with $d - c' \le \ell - 2,\quad e < 2\ell$.

$S_4 = \{0,\ 10_{e=d'},\ 14_d,\ 20_c \rightarrow\}$

$\ell = 5,\ t = 0,\ S_4$ is non-acute with $d - c' \le \ell - 2,\ e = 2\ell$.

(3) When $\widetilde{s} \le d'$ we can have every case $(a), (b), (c)$ of (2.6.4):

$S_5 = \{0,\ 13_e,\ 15_{d'},\ 20_d,\ 26_c \rightarrow\} : \ell = t = 5\quad e < 2\ell + t = 15;$

$S_6 = \{0,\ 15_e,\ 19_{d'=\widetilde{s}},\ 24_d,\ 30_c \rightarrow\} : \ell = t = 5\quad e = 2\ell + t = 15;$

$S_7 = \{0, 26_e,\ 28,\ 31_{d'},\ 33_d,\ 39_c \rightarrow\} : \ell = t = 5\quad e > 2\ell + t = 15.$

3 General results on s_m.

As seen in [6], $s_m = \widetilde{s} + d$ when $\widetilde{s} \ge d'$. To give estimations of s_m in the remaining cases we shall use the same tools as in [6]: we recall them for the convenience of the reader. We shall add some improvement, as the general inequalities (3.1.3) on the difference $\nu(s+1) - \nu(s)$, however a great part of the following (3.1),(3.3),(3.4) is already proved in [6, 4.1, 4.2, 4.3].

Proposition 3.1 *Let $S' = \{s \in S, \mid s \le d'\}$. For $s_i \in S$, let $\eta(s_i),\ \alpha(s_i),\ \beta(s_i),\ \gamma(s_i),\ \delta(s_i)$ be as in (2.1). Then:*

(1) *If $\widetilde{s} < d'$, we have:*
$s_{i+1} = s_i + 1$ *for* $s_i \ge \widetilde{s} + d' - \ell$, *in particular for* $s_i \ge 2d'$.

(2) *For each $s_i \in S$:* $\eta(s_i) = \alpha(s_i) + \beta(s_i) + \gamma(s_i) + \delta(s_i)$. *Further*

$$\alpha(s_i) = \left[\begin{array}{lll} -2 & if & (s_{i+1} - c' \notin S' \ and\ \ s_i - d \in S') \\ \ \ 0 & if & (s_{i+1} - c' \in S' \Longleftrightarrow s_i - d \in S') \\ \ \ 2 & if & (s_{i+1} - c' \in S' \ and\ \ s_i - d \notin S'). \end{array}\right.$$

$$\beta(s_i) = \left[\begin{array}{lll} \ \ 0 & if & s_i \le 2c' - 2 \ \ or\ \ s_i > 2d \\ \ \ 1 & if & 2c' - 1 \le s_i \le c' + d - 1 \\ -1 & if & c' + d \le s_i \le 2d. \end{array}\right.$$

$$\gamma(s_i) = \left[\begin{array}{lll} \ \ 0 & if & s_i \ge 2d' + 1 \\ -1 & if & s_i = 2d' \\ -1 & if & s_i < 2d' \ \ and\ \ [s_i - d', d'] \cap \mathbb{N} \subseteq S. \end{array}\right.$$

$$\delta(s_i) = \left[\begin{array}{lll} 0 & if & s_{i+1} - c \notin S,\ \ s_i \le 2c - 1 \\ 2 & if & s_{i+1} - c \in S,\ s_i \le 2c - 1 \\ 1 & if & s_i \ge 2c\ . \end{array}\right.$$

(3) *Let* $s = 2d - k < 2d$ *and* $s + 1 \in S$, *then:*

$(a)\ -\left[\dfrac{k}{2}\right]-1 \le \nu(s+1)-\nu(s) \le \left[\dfrac{k+5}{2}\right].$

(b) *If* $s = 2d' - h < 2d'$, *then* $-\left[\dfrac{h}{2}\right]-1 \le \gamma(s) \le \left[\dfrac{h+1}{2}\right].$

Proof. (1) By assumption and by (2.6.2) we have $c-e \le \widetilde{s} < d'$ and so $d'-\ell \in S$. It follows $d'-\ell \ge e$ because $d' \ge e \ge \ell+t+1$ (2.6.2). Hence $s \ge \widetilde{s}+d'-\ell \Longrightarrow s \ge c$.
(2) By [6, (3.3)...(3.7)] we have only to prove the last two statements for γ.
Let $s = 2d'-h \in S$, $h \in \mathbb{N}$, $s+1 = 2d'-h+1$ and assume $[d'-h, d'] = [s_i - d', d'] \cap \mathbb{N} \subseteq S$. Then:

$$C(s_i) = \{(d'-h, d'), (d'-h+1, d'-1), (d'-h+2, d'-2), ..., (d'-1, d'-h+1), (d', d'-h)\}$$
$$C(s_{i+1}) = \{(d'-h+1, d'), (d'-h+2, d'-1), .., (d'-1, d'-h+2), (d', d'-h+1)\}$$

it follows that $\gamma(s_i) = \#C(s_{i+1}) - \#C(s_i) = h-(h+1) = -1$.

(3.a) To prove the inequalities for $s = 2d-k$, let $2d-k = x+y$, with $x \ge y$: then $\begin{cases} y \le d - \left[\frac{k}{2}\right] \\ x \ge d - \left[\frac{k}{2}\right] \end{cases}$.

Therefore divide the interval $[d-\left[\dfrac{k}{2}\right], d] \cap \mathbb{N}$ in subsets

$$\Lambda_j = [* * * \, a_j \longleftrightarrow b_j] = H_j \cup S_j, \quad j = 1, ..., j(s)$$

with $S_j \subseteq S$, $S_j = [a_j, b_j] \cap \mathbb{N}$ interval such that $b_j + 1 \notin S$, and $H_j \subseteq \mathbb{N} \setminus S$, $H_j \ne \emptyset$, if $j > 1$ (i.e. $a_{j-1} \notin S$ for $j > 1$, $H_1 = \emptyset \Longleftrightarrow a_1 = d - [k/2] \in S$).
Let $N_j(s) := N(s) \cap \{(x,y), (y,x) \mid x \in S_j,\ x \ge y\}$: we have $N(s) = \bigcup_j N_j(s) \cup D(s)$. Hence:

$$\nu(s+1) - \nu(s) = (\textstyle\sum_j n_j) + \delta(s), \text{ where } n_j = \#N_j(s+1) - \#N_j(s).$$

Further: $-2 \le n_j \le 2$. This fact follows by the same argument used to prove the formulas for $\alpha(s_i)$, $\beta(s_i)$ recalled in statement (2) above. Since $0 \le \delta(s) \le 2$ (see (2) above) we conclude that

$$(*) \qquad -2j(s) \le \nu(s+1) - \nu(s) \le 2j(s) + 2.$$

More precisely, to evaluate the largest and lowest possible values of $\nu(s+1)-\nu(s)$, with $s = 2d-k$, we consider separately four cases: $\left[\begin{array}{ll} (A) & k = 4p \\ (B) & k = 4p+1 \\ (C) & k = 4p+2 \\ (D) & k = 4p+3. \end{array}\right.$

In each case we can see that $j(s) \le p+1 = \left[\dfrac{k}{4}\right]+1$. First note that $d \in S$, hence $j(s)$ is maximal when $\#H_j = \#S_j = 1$, i.e. $[d - \left[\frac{k}{2}\right], d] = [... * \times * \times ... * d]$ (where $\times$ means element in S).
In each of the above cases we shall find integers x_1, x_2, y_1, y_2 such that $\begin{cases} x_1 \le \nu(s+1) \le x_2 \\ y_1 \le \nu(s) \le y_2 \end{cases}$, then the statement will follow by the obvious inequality $x_1 - y_2 \le \nu(s+1) - \nu(s) \le x_2 - y_1$.

- If either $k = 4p$, or $k = 4p+1$, then $j(s)$ is maximal if and only if

$$[d - \left[\tfrac{k}{2}\right], d] = [d - 2p * ... * d - 2 * d], \text{ with } j(s) = p+1.$$

Note that when $j(s) = p+1$, then $1 \le \#\big(N(s) \setminus D(s)\big) \le 2p+1$ because $(d-2p, d-2p) \in N(s)$; further we have $p \le j(s+1) \le p+1$ and so $0 \le \#N(s+1) \le 2p+4$.
If $k = 4p$, we have $1 \le \#\big(N(s) \setminus D(s)\big) \le 2p+1$, since $(d-2p, d-2p) \in N(s)$, further $j(s+1) = p$, hence $0 \le \#\big(N(s+1) \setminus D(s+1)\big) \le 2p$.

$$-\left[\frac{k}{2}\right] - 1 = -2p - 1 \le \nu(s+1) - \nu(s) \le 2p + 2 - 1 < \left[\frac{k+5}{2}\right].$$

If $k = 4p+1$, we have $0 \le \#\big(N(s) \setminus D(s)\big) \le 2p+2$, further $s+1 = 2d-4p$, therefore $1 \le \#\big(N(s+1) \setminus D(s+1)\big) \le 2p+1$. We obtain:

$$-\left[\frac{k}{2}\right] - 1 = -2p - 1 \le \nu(s+1) - \nu(s) \le 2p + 3 = \left[\frac{k+5}{2}\right].$$

- If either $k=4p+2$, or $k=4p+3$, then $\left[\dfrac{k}{2}\right]=2p+1$,
analogously we get $j(s)=p+1$ maximal if

$$\left[d-\left[\frac{k}{2}\right],d\right]=\left[\begin{array}{l}[\,*\,d-2p*...*d-2*d] \quad or\\ [\,d-2p-1...*\times\times*...d]\text{ (with one and only one } j_0 \text{ such that } \#S_{j_0}=2).\end{array}\right.$$

If $k=4p+2$, in the first subcase we get $0\le\big(N(s)\backslash D(s)\big)\le 2p+2$, and $0\le\#\big(N(s+1)\backslash D(s+1)\big)\le 2p$ because $(d-2p-1,d-2p)\notin N(s+1)$. Hence

$$-\left[\frac{k}{2}\right]-1=-2p-2\le\nu(s+1)-\nu(s)\le 2p+2<\left[\frac{k+5}{2}\right].$$

In the second subcase we get $1\le\#\big(N(s)\setminus D(s)\big)\le 2p+2$, because $(d-2p-1,d-2p-1)\in N(s)$ and $0\le\big(N(s+1)\setminus D(s+1)\big)\le 2p$ since $(d-2p-1,d-2p)\notin N(s+1)$. Hence

$$-\left[\frac{k}{2}\right]-1=-2p-2\le\nu(s+1)-\nu(s)\le 2p+1<\left[\frac{k+5}{2}\right].$$

If $k=4p+3$, in the first subcase we get $0\le\big(N(s)\setminus D(s)\big)\le 2p+2$, and
$0\le\#\big(N(s+1)\setminus D(s+1)\big)\le 2p+2$ because $(d-2p-1,d-2p)\notin N(s+1)$. Hence

$$-\left[\frac{k}{2}\right]-1=-2p-2\le\nu(s+1)-\nu(s)\le 2p+4=\left[\frac{k+5}{2}\right].$$

In the second subcase we get $0\le\#\big(N(s)\setminus D(s)\big)\le 2p+2$ and $0\le\#\big(N(s+1)\setminus D(s)\big)\le 2p+1$ because $(d-2p-1,d-2p-1)\in N(s+1)$. Hence

$$-\left[\frac{k}{2}\right]-1=-2p-2\le\nu(s+1)-\nu(s)\le 2p+3<\left[\frac{k+5}{2}\right].$$

$(3.b)$ The proof is quite similar to the above one: since $\gamma(s)=\#C(s+1)-\#C(s)$, we do not need to add $\delta(s)$ and so formula $(*)$ becomes

$$-2j'(s)\le\gamma(s)\le 2j'(s),$$

where $j'(s)$ is the number of the subsets Λ_j as in $(3.a)$ contained in the interval $[d'-\left[\dfrac{h}{2}\right],d']\cap\mathbb{N}$.
Then it suffices to proceed as above. $\diamond$

Example 3.2 The bounds found in $(3.1.3a)$ are both sharp. To see this fact, consider $S=\{0,10_e,20_{d'},30_d,40_c\rightarrow\}$ and the elements $s=2d-1=59$, $s+1=2d=60$. By a direct computation we easily get: $\nu(s+1)-\nu(s)=3=\left[\dfrac{k+5}{2}\right]$ (with $k=1$), and
$\nu(s+2)-\nu(s+1)=-\left[\dfrac{k}{2}\right]-1$ (with $k=0$).

Proposition 3.3 *Let $\bigcirc$ mean $\notin S'$ and $\times$ mean $\in S'$ (recall that for $s\le d'$, we have $s\in S\iff s\in S'$). The following tables describe the difference $\eta(s_i)=\nu(s_{i+1})-\nu(s_i)$ for $s_i\in S$, $s_i<2c$ in function of $\alpha,\beta,\gamma,\delta$.*

(a) *If $s_i<2c$:*

$s_{i+1}-c$	s_i-d	$s_{i+1}-c'$	α	δ	$\eta(s_i)$
$\notin S$	$\times$	$\bigcirc$	-2	0	$\beta+\gamma-2$
$\notin S$	$\bigcirc$	$\bigcirc$	0	0	$\beta+\gamma$
$\notin S$	$\times$	$\times$	0	0	$\beta+\gamma$
$\notin S$	$\bigcirc$	$\times$	2	0	$\beta+\gamma+2$
$\in S$	$\times$	$\bigcirc$	-2	2	$\beta+\gamma$
$\in S$	$\bigcirc$	$\bigcirc$	0	2	$\beta+\gamma+2$
$\in S$	$\times$	$\times$	0	2	$\beta+\gamma+2$
$\in S$	$\bigcirc$	$\times$	2	2	$\beta+\gamma+4$

More precisely we have the following subcases.

(*b*) *If* $s_i \leq 2d' - 1$, *then* $\beta = 0$:

$$\left[\begin{array}{ccccccc} s_{i+1} - c & s_i - d & s_{i+1} - c' & \alpha & \beta & \delta & \eta(s_i) \\ \bigcirc & \times & \bigcirc & -2 & 0 & 0 & \gamma - 2 \\ \bigcirc & \times & \times & 0 & 0 & 0 & \gamma \\ \bigcirc & \bigcirc & \bigcirc & 0 & 0 & 0 & \gamma \\ \times & \times & \bigcirc & -2 & 0 & 2 & \gamma \\ \bigcirc & \bigcirc & \times & 2 & 0 & 0 & \gamma + 2 \\ \times & \bigcirc & \bigcirc & 0 & 0 & 2 & \gamma + 2 \\ \times & \times & \times & 0 & 0 & 2 & \gamma + 2 \\ \times & \bigcirc & \times & 2 & 0 & 2 & \gamma + 4 \end{array}\right].$$

(*c*) *If* $s_i = 2d'$, *then* $\beta = 0, \quad \gamma = -1$:

$$\left[\begin{array}{ccccccc} s_{i+1} - c & s_i - d & s_{i+1} - c' & \alpha & \beta & \delta & \eta(s_i) \\ \bigcirc & \times & \bigcirc & -2 & 0 & 0 & -3 \\ \bigcirc & \times & \times & 0 & 0 & 0 & -1 \\ \bigcirc & \bigcirc & \bigcirc & 0 & 0 & 0 & -1 \\ \times & \times & \bigcirc & -2 & 0 & 2 & -1 \\ \bigcirc & \bigcirc & \times & 2 & 0 & 0 & 1 \\ \times & \bigcirc & \bigcirc & 0 & 0 & 2 & 1 \\ \times & \times & \times & 0 & 0 & 2 & 1 \\ \times & \bigcirc & \times & 2 & 0 & 2 & 3 \end{array}\right].$$

(*d*) *If* $s_i \in [2d' + 1, c' + d - 1]$, *then* $\beta \in \{0, 1\}, \quad \gamma = 0$:
$\nu(s_{i+1}) < \nu(s_i)$ *if and only if the following row is satisfied*

$$\left[\begin{array}{ccc} s_{i+1} - c & s_i - d & s_{i+1} - c' \\ \bigcirc & \times & \bigcirc \end{array}\right].$$

(*e*) *If* $s_i \in [c' + d, 2d]$, *then* $\beta = -1, \ \gamma = 0, \ s_i - d \in S \setminus S', s_{i+1} - c' \notin S'$, *then*
$\nu(s_{i+1}) < \nu(s_i) \iff s_i - \ell - d \notin S$.

The next theorem collects the results [6, Th. 4.1, Th.4.2, Th. 4.4] with some upgrades: statement (1) improves [6, Th.4.2], the last part of (5) is new.

Theorem 3.4 *With Setting 2.1, the following inequalities hold.*

(0) *If* $\widetilde{s} \geq 2d' - d$, *then* $s_m \leq \widetilde{s} + d$;
if $\widetilde{s} < 2d' - d$, *then* $s_m \leq 2d'$. *More precisely*

(1) *If* $\widetilde{s} \geq d' + c' - d$, *then* $s_m = \widetilde{s} + d$.

(2) *If* $\widetilde{s} = d' + c' - d - 1$, *then* $s_m \leq \widetilde{s} + d - 1$.

(3) *If* $2d' - d < \widetilde{s} < d' + c' - d - 1$, *let*

$U := \{\sigma \in [2d' + 1 - d, \ \widetilde{s}] \cap S \ \mid \ \sigma - \ell \notin S, \ \sigma + d + 1 - c' \notin S\}$:

(*a*) *if* $U \neq \emptyset$, *then* $s_m = d + max\, U$,
in particular $s_m = \widetilde{s} + d \iff \widetilde{s} + d + 1 - c' \notin S$,

(*b*) *if* $U = \emptyset$, *then* $s_m \leq 2d'$.

(4) *If* $\widetilde{s} = 2d' - d$, *then* $s_m = \widetilde{s} + d$.

(5) *If* $\widetilde{s} < 2d' - d$, *then* $s_m \leq 2d'$, *more precisely:*

$s_m = 2d' \iff 2d'$ *satisfies either row 3 or row 4 of Table 3.3 (c).*

In the case $\widetilde{s}+d+1-c' \notin S$:

(*a*) *if* $2d'-d-2 \le \widetilde{s} \le 2d'-d-1$, *then* $\widetilde{s}+d \le s_m \le 2d'$

(*b*) *if* $\widetilde{s}=2d'-d-j$, $j=3,4$ *and* $\{d'-j,...,d'-1\} \cap S \neq \{d'-j+1\}$ *then* $s_m \ge \widetilde{s}+d$.

Proof. (0) is proved in [6, (4.4.1),(4.4.3)].
Now recall that $\widetilde{s} \ge c-e$ (2.6.2), hence $\widetilde{s}+d+1 \ge c+1 \in S$; further in cases (1) and (2) $\widetilde{s}+d+1-c' \ge d'$, hence (1) and (2) follow by (0) and by Tables 3.3 (d), (e).
The cases (3) and (4) follow easily by Tables 3.3 (d) and (c).
(5) We have $s_m \le 2d'$ by (0); further $2d'$ cannot satisfy the first two rows of Table 3.3 (c) since $\widetilde{s} < 2d'-d$.
By a direct computation we can see that we always have $\gamma(2d'-j) \le 1$, for $j \le 2$, while for $j=3,4$ $\gamma(2d'-j) \le 1 \Longleftrightarrow \{d'-j,...,d'-1\} \cap S \neq \{d'-j+1\}$. Now ($a$) and ($b$) follow because $\nu(\widetilde{s}+d) > \nu(\widetilde{s}+d+1)$ by Table 3.3.(b). ⋄

The following conjecture gives a lower bound for s_m, it is justified by calculations in very many examples. We are able to prove that it holds in many cases.

Conjecture 3.5 *For every semigroup the inequality* $s_m \ge c+d-e$ *holds.*

First we note that (3.5) holds in the following general cases:

Proposition 3.6 *Assume* $\left[\begin{array}{ll} \textit{either} & s_m \ge \widetilde{s}+d \\ \textit{or} & s_m \ge 2d' \ \textit{and} \ \widetilde{s} < d' \end{array}\right.$. *Then* $s_m \ge c+d-e$.

In particular if either $\widetilde{s} \ge c'+d'-d$ *or* $\widetilde{s}+d=2d'$, *then* $s_m \ge c+d-e$.

Proof. The first statement follows by (2.6.2): in fact $\widetilde{s} \ge c-e$.
If $\widetilde{s} < d'$, $s_m \ge 2d'$, we have $d' \ge c-e+\ell$ (2.6.1b) and $d-d' \le \ell$ (2.8.1). Hence $s_m \ge 2d' \ge d'+c-e+\ell \ge c+d-e$. Now the particular cases follow by (3.4.1,4). ⋄

Corollary 3.7 (1) *If* $s_m > 2d'$, *then* $s_m-d \in S$.

(2) *If* $\widetilde{s}=d'-1$, *then* $s_m=\widetilde{s}+d \Longleftrightarrow c' \neq d$.

Proof. (1) follows by Table 3.3.(d).
(2) If $c'=d$, then $s_m \neq \widetilde{s}+d$, by (3.4.2).
If $c' \neq d$ and $\widetilde{s}=d'-1$, we have $\widetilde{s} \ge d'+c'-d$ then apply (3.4.1). ⋄FFFFFFFFFFFFFF

Proposition 3.8 *Assume* $\widetilde{s} \le d'-2$ *and* $[\widetilde{s}+2, d'] \cap \mathbb{N} \subseteq S$. *Then* $s_m \le \widetilde{s}+d$:

(1) *if* $2d'-d < \widetilde{s} < d'+c'-d$, *then* $s_m \left[\begin{array}{l} =\widetilde{s}+d \Longleftrightarrow \widetilde{s}+1 \notin S \textit{ and } c'=d \\ \le \widetilde{s}+d-1, \quad \textit{otherwise}. \end{array}\right.$

(2) *if* $\widetilde{s} \le 2d'-d$, *then* $s_m=\widetilde{s}+d$.

Proof. In case (1), by applying Theorem 3.4 we see that $s_m \le \widetilde{s}+d$; further $s_m=\widetilde{s}+d \Longleftrightarrow \widetilde{s}+d+1-c' \notin S$. Since $\widetilde{s}+1 \le \widetilde{s}+d+1-c' \le d'$ by the assumptions, we see that $s_m=\widetilde{s}+d \Longleftrightarrow c'=d$ and $\widetilde{s}+1 \notin S$.
In case (2), by Theorem 3.4 we have $s_m \le 2d'$.
Now let $\widetilde{s}+d+1 \le s \le 2d'$. Then by the assumptions we get

$$\begin{cases} \widetilde{s}+2 \le s+1-c' \le s-d'-1 < d' \\ s+1 \in S \ \text{ and } \ s+1-c' \in S' \\ \{s-d',...,d'\} \subseteq S \ \text{ (hence } \gamma(s)=-1 \text{ (3.3.2))} \\ s-\ell-d \in S \ \text{ (by (2.4.1))}. \end{cases}$$

From Tables 3.3 (b) − (c) we conclude that $s_m < s$ and also that $s_m=\widetilde{s}+d$; in fact $\widetilde{s} \in S$, $\widetilde{s}-\ell \notin S$, further $\widetilde{s}+d-d' \ge \widetilde{s}+2$, because $d-d' \ge 2$, therefore $\gamma(\widetilde{s}+d)=-1$ by the assumptions and by (3.3.2). ⋄

Remark 3.9 (1) Both situations of (3.8.1) above can happen, even for $\ell = 3$ (see the following (4.7)):
(A) If $\ell = 3,\ t = 5,\ d' = d-3,\ \ c' = d-1$, (4.7.case A) we have $s_m < \widetilde{s} + d$.
(B) If $\ell = 3,\ t = 5,\ d' = d-3,\ d-4 \notin S,\ \ \ c' = d$, (4.7.case B) we have $s_m = \widetilde{s} + d$.

(2) Assume $2d' - d < \widetilde{s} \le d' + c' - d - 1$ and $[d' - \ell + 2, d'] \cap \mathbb{N} \subseteq S$; then the set U of (3.4.3) is empty. In fact for each $s \in S$, such that $2d' + 1 \le s \le \widetilde{s} + d$, we have $s + 1 \in S$, and by (2.8.1), $d' - \ell + 2 \le 2d' + 2 - d \le s + 1 - c' \le d'$, therefore $s + 1 - c' \in S'$.

(3) If $s_m < 2d' \le \widetilde{s} + d$, then $\begin{cases} (a)\ \widetilde{s} + d + 1 - c' \in S \\ (b)\ \widetilde{s} + d + 1 - c' - \ell \in S \\ (c)\ \{2d' - d - \ell, 2d' + 1 - c'\} \cap S \ne \emptyset. \end{cases}$

(3.a) holds by (3.4.3); in fact the assumptions imply $U = \emptyset$ because $\widetilde{s} - \ell \notin S$.
(3.b) is clear by (3.a) and by (2.4.1) , since $\widetilde{s} < \widetilde{s} + d + 1 - c' < d$.
(3.c) follows by Table 3.3 (c)).

(4) The assumption $s_m > 2d'$ in (3.7.1) is necessary: for instance if $S = \{0, 20_e, 21, 26, 27_{d'}, 32_d, 39_c \rightarrow\}$ we have $s_m = 2d'$, with $2d' - d \notin S$ (we deduce $s_m = 2d'$ by Table 3.3 (c)).

Proposition 3.10 *If* $\widetilde{s} < d'$ *and* $[d' - \ell, d'] \cap \mathbb{N} \subseteq S$, *let* $h = d - c',\quad q = d - d'$. *Then*

(1) $[\widetilde{s} - \ell + 1, d'] \cap \mathbb{N} \subseteq S$ *and* $e \ge 2\ell + t$.

(2) *If* $2d' - d < \widetilde{s} \le d' + c' - d - 1$, *we have*

(a) $q + h + 1 \le t < 2q\ (\le 2\ell)$,

(b) *For* $s \in [\widetilde{s} + d' - \ell + 1, 2d'] \cap S$, *we have* $\gamma(s) = -1$.

(c) *We have* $s_m \le 2d'$.

(d) *Let* $W := [\widetilde{s} - 2\ell + 1, 2d' - \ell - d] \cap \mathbb{N} \setminus S$.
If $W \ne \emptyset$, *let* $h_0 := max\ W$, *then* $s_m \ge h_0 + \ell + d$.

(e) $s_m < \widetilde{s} + d - \ell + 1 \Longleftrightarrow [\widetilde{s} - 2\ell + 1, 2d' - d - \ell] \cap \mathbb{N} \subseteq S$,
$s_m < \widetilde{s} + d - \ell + 1 \Longrightarrow e \ge 3\ell + t$.

(f) *If* $[\widetilde{s} - 2\ell + 1, 2d' - d - \ell] \cap \mathbb{N} \subseteq S$, *then* $s_m \ge \widetilde{s} + d' - \ell + 1$.

Proof. (1) By the assumptions and by (2.4.1) we have $[\widetilde{s} - \ell + 1, d'] \cap \mathbb{N} \subseteq S$; the inequality $e \ge 2\ell + t$ follows by (2.6.3)
(2) Statement (a) is immediate by the assumption $2d' - d < \widetilde{s} \le d' + c' - d - 1$.
(b) follows by (1) and by (3.3.2).
(c) By (3.4.3) we know that $s_m \le \widetilde{s} + d$. For each $2d' < s \le \widetilde{s} + d$ we have $d' - \ell < 2d' + 1 - c' < s + 1 - c' \le \widetilde{s} + d + 1 - c' \le d' + c' - 1 + 1 - c' = d'$. Therefore $s + 1 - c' \in S$ and $s_m \le 2d'$ by (3.4.3b).
(d) and (e) Note that $s \in [\widetilde{s} + d - \ell + 1, 2d'] \cap S \Longrightarrow \widetilde{s} - \ell + 1 \le s - d \le s + 1 - c' \le s - d' \le d'$, hence $\{s - d,\ s + 1 - c'\} \subseteq S'$ and $\gamma(s) = -1$, by (b) and the assumptions. By Table 3.3 (b) we get

$$\nu(s) > \nu(s+1) \Longleftrightarrow s + 1 - c \notin S.$$

Then (d) follows and the equivalence (e) becomes immediate by (c) and (d), recalling that $s + 1 - c = s - \ell - d$. We get $e \ge 3\ell + t$ by (2.6.1−2), since $d - (2\ell + t - 1) \in S$ and $2\ell + t - 1 < e$ by (1).
(f) For $s \in [\widetilde{s} + d' - \ell + 1, 2d' - \ell] \cap \mathbb{N}$, we have $\gamma(s) = -1$ (see (b)). If there exists $\overline{s} \in [\widetilde{s} + d' - \ell + 1, 2d' - \ell] \cap \mathbb{N}$, $\overline{s} + 1 - c' \notin S$, we have $\overline{s} - d \in S'$ by the assumptions and so $s_m \ge \overline{s}$ by Table 3.3 (b); the claim follows.
Assume on the contrary that $[\widetilde{s} + d' - \ell + 2 - c', 2d' - \ell + 1 - c'] \cap \mathbb{N} \subseteq S$: then

$$[\widetilde{s} - 2\ell + 1, 2d' - \ell + 1 - c'] \cap \mathbb{N} \subseteq S.$$

In fact $q + h + 1 \le t$ (3.10.1) $\Longrightarrow \widetilde{s} + d' - \ell + 2 - c' = d - q - \ell - t + 2 + h \le d - 2q - \ell + 1 = 2d' - \ell + 1 - d$.
We can iterate the algorithm looking for one element $\overline{\overline{s}} \in [2d' - \ell + 1, 2d' - \ell + d + 1 - c'] \cap \mathbb{N}$ such that $\overline{\overline{s}} + 1 - c' \notin S$. If needed we repeat the argument till we find s' such that $s' + 1 - c' \notin S$: s' surely

exists since $\widetilde{s} - \ell \notin S$. $\diamond$

The previous results can be summarized in the following theorem.

Theorem 3.11 *Assume* $[d' - \ell, d'] \cap \mathbb{N} \subseteq S$. *Then* $s_m \geq c + d - e$. *In particular:*

(1) *if* $2d' - d < \widetilde{s} < d' + c' - d$, *we have* $c + d - e \leq \widetilde{s} + d' - \ell + 1 \leq s_m \leq 2d'$,

(2) $s_m = \widetilde{s} + d$ *in the remaining cases.*

Proof. (1) Since $\widetilde{s} < d'$, we have $[\widetilde{s} - \ell + 1, d'] \cap \mathbb{N} \subseteq S$, $e \geq 2\ell + t$ by (3.10.1). It follows that $\widetilde{s} - \ell + 1 \geq c - e$ because $[c - \ell - e, c - e - 1] \cap S = \emptyset$ and $\widetilde{s} \geq c - e$ by (2.6.2).
The inequalities follow by items $(c,)(d), (e), (f)$ of (3.10):
if the set W of (3.10.2d) is not empty then we see that $s_m \geq \widetilde{s} + d - \ell + 1 \geq \widetilde{s} + d' - \ell + 3$ by (3.10.2d), recalling that $d' \leq d - 2$.
If $W = \emptyset$, by (3.10.2f) we get $s_m \geq \widetilde{s} + d' - \ell + 1$.
(2) follows by (3.4.1) and by (3.8.2). In this case $s_m \geq c + d - e$ by (3.6).
To prove $s_m \geq c + d - e$ in case (1), first assume that $W = \emptyset$: since $d' \geq d - \ell$ (2.6.3b) and $e \geq 3\ell + t$ (3.10.2f), we get $s_m \geq \widetilde{s} - \ell + d' + 1 \geq \widetilde{s} - \ell + d - \ell + 1 = c + d - 3\ell - t \geq c + d - e$.
If $W \neq \emptyset$, since $e \geq 2\ell + t$ we get

$$s_m \geq \widetilde{s} + d - \ell + 1 = c + d - 2\ell - t \geq c + d - e,$$

further $d > d' + 1 \Longrightarrow \widetilde{s} + d - \ell + 1 > \widetilde{s} + d' - \ell + 1$. $\diamond$

4 Some particular case.

In this section we shall estimate or give exactly the value of s_m in some particular case. Since $s_m = \widetilde{s} + d$ for each semigroup S satisfying $\widetilde{s} \geq c' + d' - d$, in this section we shall often assume $\widetilde{s} < c' + d' - d$.

4.1 Relations between the order bound and the holes set H.

Let $H := [c - e, c'] \cap \mathbb{N} \setminus S$ be as in (2.1): when H is an interval we deduce the value of s_m; if $\#H \leq 2$ and in some other situation we give a lower bound for s_m.

Proposition 4.1 (1) *If* $H = \emptyset$, *then* $c' = c - e$ *and* S *is acute with* $s_m = \widetilde{s} + d$.

(2) *Assume that* $\widetilde{s} < d'$. *Then the conditions*

> (a) $[d' - \ell, d'] \cap \mathbb{N} \subseteq S$ *and* $e = 2\ell + t$
>
> (b) $H = [d' + 1, c' - 1] \cap \mathbb{N}$
>
> *are equivalent and imply:* $s_m = \left[\begin{array}{ll} 2d' & if \;\; 2d' - d + 1 \leq \widetilde{s} \leq d' + c' - d - 1 \\ \widetilde{s} + d & in\ the\ remaining\ cases. \end{array} \right.$

Proof. (1) $H = \emptyset \Longleftrightarrow c' = c - e$; then apply (2.3. 5 and 4).
(2), $(a) \Longleftrightarrow (b)$. (a) implies that $[\widetilde{s} - \ell + 1, d'] \cap \mathbb{N} \subseteq S$, and $\widetilde{s} - \ell = c - e - 1$ (3.10.1) and (2.6.4). Hence (b) holds. On the contrary, (b) implies $[c - e, d'] \cap \mathbb{N} \subseteq S$. Since $c - e \leq d' - \ell$ by (2.6.3), we get $[d' - \ell, d'] \cap \mathbb{N} \subseteq S$, further $e = 2\ell + t$ by (2.8.3 and 4).

Now assume that $(a) - (b)$ hold. Since $c - e \leq d' - \ell$ by (2.6.1), when $2d' - d < \widetilde{s} < d' + c' - d$, we have by (2.8.1) $c - e - \ell \leq d' - \ell - (d - d') = 2d' - \ell - d < \widetilde{s} - \ell \leq c - e - 1$. We obtain $s_m \geq 2d'$ by (3.10.2d) because the set W as in (3.10.2d) has $2d' - \ell - d = max\ W$. Now $s_m = 2d'$ follows by (3.10.2c). For the statement in the remaining cases see (3.11.2). $\diamond$

Example 4.2 When $\widetilde{s} > d'$ the implication $(b) \Longrightarrow (a)$ in (4.1.2) is not true in general: in fact $S := \{0, 8_e, 12_{c-e=d'}, 14_{c'}, 15, 16_d * * * 20_c \rightarrow\}$ has $H = \{c' - 1\} = \{13\}$, $t = 0$, $\ell = 3$, $e \neq 2\ell + t$.

Proposition 4.3 *Assume* $\widetilde{s} < c' + d' - d$. *Let* $k := min\{n \in \mathbb{N} \mid d' - n \notin S\}$, $h := d - c'$, $s := d' + c' - k - 1$. *We have*

(1) $s \leq 2d' \iff c' - d' \leq k + 1 \iff d - d' \leq k + h + 1$.

(2) *If* $\widetilde{s} < d' - k$ *and* $\ell \leq k + h + 1$, *then* $s_m \geq s \geq c + d - e$.

(3) *If* $1 \leq k < \ell$, $c' - d' \leq k + 1$ *and* $\{d' - \ell, ..., d'\} \setminus \{d' - k\} \subseteq S$, *then* $c + d - e \leq s \leq s_m \leq 2d'$.

Proof. (1) is obvious by the assumptions.
(2) We have $[d' - k - \ell + 1, d' - \ell] \cap \mathbb{N} \subseteq S$ (2.4.1).
Now we claim that $\gamma(s) = -1$. In fact the assumption $\ell \leq k + h + 1$ implies $c' - d' = d - d' - h \leq \ell - h \leq k + 1$, and so $s \leq 2d'$; since $[s - d', d'] \cap \mathbb{N} \subseteq S$, then $\gamma(s) = -1$ (3.1.2). Since $\widetilde{s} < d'$, then $d - c' \leq \ell - 2$, further $\ell \leq k + h + 1$; therefore

$$d' - k - \ell + 1 \leq s - d = d' + c' - k - 1 - d \leq d' - \ell.$$

Hence $s - d \in S$. Moreover $s + 1 - c' = d' - k \notin S$. Then $s_m \geq s$ by Table 3.3 (b).
To prove that $s \geq c + d - e$, recall that $c' - d' \leq k + 1$. Then by assumption we have $\widetilde{s} < d' - k < c' - k - 1 \leq d'$. Then $c' - k - 1 \in S$ and by (2.6.3) we get $c' - k - 1 \geq c - e + \ell$. Hence

$$s = d' + c' - k - 1 \geq d - \ell + c' - k - 1 \geq c + d - e \text{ (2.8.1)}$$

(3) By assumption $\widetilde{s} < d' - h \leq d'$, and so $[d' - h - \ell, d' - \ell] \cap \mathbb{N} \setminus \{d' - k - \ell\} \subseteq S$. Hence $[d' - h - \ell, d' - k - 1] \cap \mathbb{N} \setminus \{d' - k - \ell\} \subseteq S$. Now recalling that $k < \ell$ we get:

$$d' - h - \ell \leq s - d < d' - k - h \leq d' - k.$$

Hence $s - d \in S$: in fact $s - d \neq d' - k - \ell$ because $s - d \geq d' - k - \ell + 1$. Further $s + 1 - c' \notin S$ and $\gamma(s) = -1$ by (3.1.2) since $s \leq 2d'$ and $\{s - d', ..., d'\} \subseteq S$. Then $s \leq s_m$ by Table 3.3 (b).
The inequality $s \geq c + d - e$ can be proved as in (2).
In order to prove the last inequality, by the assumptions on $\widetilde{s}$ and by (3.4) it suffices to consider elements $u \in [2d' + 1, c' + d' - 1] \cap \mathbb{N}$. For such an element u we have $d' \geq u + 1 - c' > s + 1 - c' = d' - k$; hence $u + 1 - c' \in S'$, then $\nu(u + 1) \geq \nu(u)$ by Table 3.3 (d). ⋄

Corollary 4.4 *Suppose* $\widetilde{s} < c' + d' - d$ *and* $\#H \leq 2$. *Then* $s_m \geq c + d - e$.

Proof. If $\#H = 0$, we have $s_m = \widetilde{s} + d$ and we are done by (4.1.1) and (3.6).
If ($\#H = 2$ and $H = \{d' + 1, c' - 1\}$), or $\#H = 1$, then either $s_m = \widetilde{s} + d$, or $s_m = 2d'$ (4.1.2); now see (3.6).
Finally assume that $H = \{d' - k\} \cup \{d' + 1\}$, with $k \geq 1$. In this case we have $c' - d' = 2 \leq k + 1$, since $k \geq 1$. Hence the claim $s_m \geq c + d - e$ follows by (3.11), if $k > \ell$, and by (4.3.3) if $k < \ell$. ⋄

4.2 Case $\ell = 2$.

If $\ell = 2$, the conjecture (3.5) is true, more precisely by [6, Thm 5.5] we have:

Proposition 4.5 *Assume* $\ell = 2$, *then* $s_m \geq c + d - e$ *and*

(1) $s_m = \widetilde{s} + d$ *if* $\left[\begin{array}{l} t \leq 2, \\ t = 4 \\ t \geq 5 \text{ and } d - 3 \in S. \end{array}\right.$

(2) $s_m = 2d - 4$ *if* $\left[\begin{array}{lll} \textit{either} & t = 3 & \text{and } d - 6 \notin S \\ \textit{or} & t \geq 5 & \text{and } d - 3 \notin S. \end{array}\right.$

(3) $s_m = 2d - 6$ *if* $t = 3$ *and* $d - 6 \in S$ (*all the remaining cases*).

Proof. The value of s_m is known by [6, Thm. 5.5]. Another proof can be easily deduced by (2.4.1), (3.4.2), (4.3.3), (3.8.2), Table 3.3 (d), (3.10. d, f). The inequalities $s_m \geq c + d - e$ now follow respectively by (3.6) and by (3.11.3). ⋄

4.3 Case $\ell = 3$.

If $\ell = 3$, we compute explicitly the possible values of s_m and we show that the conjecture (3.5) holds.

Notation 4.6 (1) If $s_i = 2d - k \in S$, $k \in \mathbb{N}$, let

$$M(s_i) := \{(s_h, s_j) \in S^2 \mid s_i = s_h + s_j, s_h \leq d, s_j \leq d\}.$$

Note that $M(s_i) = \{(d-x, d-y) \in S^2, \mid 0 \leq x, y \leq k, \ x+y=k\}$ and that for $s_i \geq c$, we have $s_{i+1} - c = d - \ell - k$; for short it will be convenient to use the following notation.

$$(*) \quad \begin{cases} \Sigma := \{z \in \mathbb{N}, \mid z \leq d, \ d - z \in S\} \\ (c, h) \in S \times \Sigma, \ h = d + c - s_i \text{ instead of the pair } (c, s_i - c) \in N(s_i). \end{cases}$$

If $\ell = 3$, then $e \geq t + \ell + 1 = t + 4$ (2.6.2). To calculate the value of s_m, we shall assume $\widetilde{s} < c' + d' - d$, otherwise $s_m = \widetilde{s} + d$ by (3.4.1). Then we have $t \geq 3$, $d - 3 \in S$ and $d - 3 \leq d'$, by (2.5.2) and by (2.8.31). Three cases are possible:

$$\begin{array}{lll} \textit{Case A}: & S = \{0, e, ..., d-3, *, d-1, d, ***, c = d+4 \rightarrow\} & d' = d-3 \\ \textit{Case B}: & S = \{0, e, ..., d-3, * \ \ *, d, ***, c = d+4 \rightarrow\} & d' = d-3 \\ \textit{Case C}: & S = \{0, e, ..., d-3, d-2, *, d, ***, c = d+4 \rightarrow\} & d' = d-2. \end{array}$$

To describe $M(2d-k)$ we shall use the notations $(*)$ fixed in (4.6) and forwhen necessary for an element $2d - k$ we shall list all the pairs $(x, y) \in M'(2d-k)$ and the pair $(c, \ell + k + 1) \in S \times \Sigma$ (the pairs underlined $\underline{(\ , \ )}$ not necessarily belong to Σ^2).

Proposition 4.7 *Assume $\ell = 3$. Then $s_m \geq c + d - e$. More precisely the values of s_m can be computed as follows.*

Case A. We have: $$s_m = \left[\begin{array}{lll} \widetilde{s} + d & if & either\ t \in [0,7] \setminus \{5\} \ \ or \ \ (t \geq 8, d-5 \in S) \\ 2d-7 & if & t \geq 8, \ d-5 \notin S \\ & if & t = 5: \\ 2d-6 & if & d-9 \notin S \\ 2d-7 & if & d-9 \in S, d-10 \notin S \\ 2d-9 & if & \{d-9, d-10\} \subseteq S, \ d-12 \notin S \\ 2d-10 & if & \{d-9, d-10, d-12\} \subseteq S \end{array} \right.$$

Proof. $S = \{0, e, ..., d-3, *, d-1, d, ***, c = d+4 \rightarrow\}$, with $e \geq \ell + t + 1 = t + 4$ (2.6.1).
First we have $s_m = \widetilde{s} + d$ if $t \leq 2d - c' - d' = 4$ and $s_m < \widetilde{s} + d$, if $t = 5$ by (3.4.1 and 2). Hence we can assume $t \geq 5$, so that $d - 4 = d - 1 - \ell \in S$, $d - 3 - \ell = d - 6 \in S$, i.e., $\{0, 1, 3, 4, 6\} \subseteq \Sigma$.

If $t = 5$ we have $[d' - \ell, d'] \cap \mathbb{N} \subseteq S$ and $2d' = 2d - 6 < \widetilde{s} + d = d' + c' - 1$. We obtain that $2d - 10 \leq s_m \leq 2d'$, $e \geq 2\ell + t$ and $s_m \geq c + d - e$ by (3.11). More precisely we can verify that:

$$s_m = \left[\begin{array}{lll} 2d-6 & if\ 9 \notin \Sigma & s_m = 2d' \geq c + d - e \\ 2d-7 & if\ 9 \in \Sigma \ and\ 10 \notin \Sigma & s_m = c + d - 11 \geq c + d - e \\ 2d-9 & if\ 9 \in \Sigma, \ 10 \in \Sigma, 12 \notin \Sigma & s_m = c + d - 13 \geq c + d - e \\ & & in\ fact\ 10 \in \Sigma \Longrightarrow e \geq 14 \\ 2d-10 & if \{9, 10, 12\} \subseteq \Sigma & s_m \geq c + d - e. \end{array} \right.$$

Note that in this case we have $d + d' - \ell - t + 1 = 2d - 2\ell - t + 1 = 2d - 10$ and this bound is achieved if $\{9, 10, 12\} \subseteq \Sigma$ (with $e \geq 16$). See, e.g. $S = \{0, 16, 34, 36, 37, 39, 40_{\widetilde{s}}, 41, 42, 43_{d'}, 45, 46_d, 50_c \rightarrow\}$.

If $t \geq 6$ we have $\widetilde{s} \leq 2d' - d$ and we consider the following subcases.

If $t = 6$, then $\widetilde{s} = 2d' - d = s_m$ by (3.4.4).

If $t \geq 7$ and $5 \in \Sigma$, one has $[d' - \ell, d'] \cap \mathbb{N} \subseteq S$ and $\widetilde{s} < 2d' - d$, hence $s_m = \widetilde{s} + d$, by (3.11).

If $t \geq 7$ and $5 \notin \Sigma$, we know that $s_m \leq 2d'$ by (3.4.5); one can compute directly that

$$\nu(2d') < \nu(2d'+1) \text{ (see Table 3.3 } (c)) \text{ and that } \nu(2d-7) > \nu(2d-6),$$

hence $s_m = 2d - 7 = 2d' - 1$. Since $d - 6 = d' - \ell \in S$, we get $e \geq 2\ell + 1 + 6 = 13$ (2.6.3), and so $s_m \geq c + d - e + 2$.

Case B. We have: $$\left[\begin{array}{ll} Case \;\; t\le 3: & s_m=\widetilde{s}+d \\ Case \;\; t\ge 4,\;\; d-5\notin S: & s_m=2d-6 \\ Case \;\; t\ge 4,\;\; d-5\in S, d-4\in S: & \\ \quad if \;\; t\in\{4,5\}, & s_m\in[2d-9,2d-6] \\ \quad if \;\; t\ge 6, & s_m \;=\widetilde{s}+d \\ Case \;\; t\ge 5,\;\; d-5\in S,\;\; d-4\notin S: & \\ \quad if \;\; t\in\{5,6,8\}, & s_m \;=\widetilde{s}+d \\ \quad if \;\; t=7, & s_m\in\{2d-8,2d-11\} \\ \quad if \;\; t\ge 9,\;\; d-7\notin S, & s_m=2d-8 \\ \quad if \;\; t=9,\;\; d-7\in S, & s_m\in\{2d-10,2d-11,2d-13\} \\ \quad if \;\; t\ge 10,\;\; d-7\in S, & s_m=\widetilde{s}+d. \end{array}\right.$$

Proof. $S=\{0,e,...,d-3,*\;\;*\;,d,***,c=d+4\rightarrow\},\quad e\ge t+4.$
As in case A we can see that $s_m=\widetilde{s}+d$ if $t\le 3$ and $s_m<\widetilde{s}+d$ for $t=4$. Suppose $t\ge 4$. Then $\{0,3,6\}\subseteq\Sigma$. We deduce the statement by means of the following table:

$$\left[\begin{array}{lll} 2d-3 & (0,3) & \\ 2d-4 & \underline{(0,4)} & \underline{(c,8)} \\ 2d-5 & \underline{(0,5)} & \underline{(c,9)} \\ 2d-6 & \underline{(0,6)}(3,3) & \underline{(c,10)}. \end{array}\right.$$

If $4\in\Sigma,\ 5\in\Sigma$, we have $[d'-\ell,d']\cap\mathbb{N}\subseteq S$ and by applying (3.11) we get $s_m\ge c+d-e$. More precisely, one can easily verify that for $t\in\{4,5\}$ we have $2d-9\le s_m\le 2d-6$, for $t\ge 6$ we have $\widetilde{s}\le 2d'-d$, then by (2.4.1) and by (3.8.2) we get $s_m=\widetilde{s}+d$.

If $5\notin\Sigma$, we have $s_m=2d-6$.

If $4\notin\Sigma,\ 5\in\Sigma$:
we have $s_m=\left[\begin{array}{lll} 2d-5 & \Longleftrightarrow & t=5 \\ 2d-6 & \Longleftrightarrow & t=6\ (8\in\Sigma,\ 9\notin\Sigma); \end{array}\right.$

the remaining cases to consider satisfy $\{0,3,5,6,8,9\}\subseteq\Sigma, 4\notin\Sigma$, with $t\ge 7,\ \ s_m<2d-6$:

$$\left[\begin{array}{lll} 2d-7 & \underline{(0,7)} & \underline{(c,11)} \\ 2d-8 & (0,8)(3,5) & \underline{(c,12)}\;\; s_m=2d-8\Longleftrightarrow \left[\begin{array}{l} 7\notin\Sigma\;\; or \\ 7\in\Sigma, 11\notin\Sigma\ (\implies 7\le t\le 8) \end{array}\right. \\ & otherwise\;\; 7,11\in\Sigma: & \{0,3,5,6,7,8,9,11\}\subseteq\Sigma, 4\notin\Sigma \\ 2d-9 & (0,9)(3,6) & \underline{(c,13)} \\ 2d-10 & \underline{(0,10)}(3,7)(5,5) & \underline{(c,14)}\;\; s_m=2d-10\Longleftrightarrow 10\in\Sigma,\ 13\notin\Sigma(\implies 9\le t\le 10) \\ & otherwise & \left[\begin{array}{lll} either & (\alpha) & 10,13\in\Sigma \\ or & (\beta) & 10\notin\Sigma\ (t=7) \end{array}\right. \end{array}\right.$$

Case (α): $\{0,3,5,6,7,8,9,10,11,13\}\subseteq\Sigma, 4\notin\Sigma\ \ (\implies t\ge 9)$:

$$\left[\begin{array}{lll} 2d-10 & (0,10)(3,7)(5,5) & \underline{(c,14)} \\ 2d-11 & (0,11)(3,8)(5,6) & \underline{(c,15)}\;\; s_m=2d-11\Longleftrightarrow 14\notin\Sigma \\ & & (\implies t=9\ if\ 12\notin\Sigma,\ t=11\ if\ 12\in\Sigma) \\ & otherwise\;\; 14\in\Sigma: & \{0,3,5\longleftrightarrow 11,13\}\subseteq\Sigma, 4\notin\Sigma \\ 2d-12 & \underline{(0,12)}(3,9)(5,7)(6,6) & (c,16)\;\; s_m=2d-12\Longleftrightarrow t=12 \\ & otherwise & \left[\begin{array}{lll} either & (\alpha 1) & 12\notin\Sigma(\Longleftrightarrow t=9) \\ or & (\alpha 2) & 12\in\Sigma,\;\; 15\in\Sigma \end{array}\right. \\ 2d-13 & (0,13)(3,10)(5,8)(6,7) & \underline{(c,17)}\;\; s_m=2d-13\left[\begin{array}{l} in\ case\ (\alpha 1) \\ in\ case\ (\alpha 2)\Longleftrightarrow t=13 \end{array}\right. \\ & otherwise\ 16\in\Sigma & \{0,3,5\longleftrightarrow 16\}\subseteq\Sigma\ 4\notin\Sigma: \\ 2d-14 & (0,14)(3,11)(5,9)(6,8)(7,7) & \underline{(c,18)}\;\; s_m=2d-14\Longleftrightarrow t=14 \\ & otherwise\ 17\in\Sigma, & \{0,3,5\longleftrightarrow 15,16,17\}\subseteq\Sigma,\ 4\notin\Sigma... \end{array}\right.$$

Clearly in cases $(\alpha 2)$, for each $t\ge 13$ we get $s_m=\widetilde{s}+d$.

Case (β): $\{0,3,5,6,7,8,9,11,\} \subseteq \Sigma,\ 4,10 \notin \Sigma$ $(t=7)$:

$$\left[\ 2d-11 \quad (0,11)(3,8)(5,6) \quad \underline{(c,15)} \quad s_m = 2d-11.\right.$$

Case C. We have: $s_m = \left[\begin{array}{lll} & if & t=3: \\ 2d-4 & if & d-4\in S,\ d-7\notin S \\ 2d-5 & if & (\{d-4,d-7\}\subseteq S,\ d-8\notin S)\ or\ (d-4\notin S) \\ 2d-7 & if & \{d-4,d-7,d-8\}\subseteq S \\ & if & t\geq 4: \\ \widetilde{s}+d & if & \quad d-4\in S \\ 2d-5 & if & \quad d-4\notin S. \end{array}\right.$

Proof. $S=\{0,e,...,d-3,d-2,*,d,***,c=d+4\to\}, d'=d-2$.
As above we see that $t\geq 3,\ \ d-3,d-5\in S$ (i.e. $\{0,2,3,5\}\subseteq\Sigma$), $e\geq 7$. Consider the table:

$$\left[\begin{array}{llllll} 2d-2 & (0,2) & & & & \\ 2d-3 & (0,3) & \underline{(c,7)} & & & \\ 2d-4 & \underline{(0,4)}(2,2) & \underline{(c,8)} & s_m = & 2d-4 & if\ 4\in\Sigma, 7\notin\Sigma \\ 2d-5 & (0,5)(2,3) & \underline{(c,9)} & s_m = & 2d-5 & if \left[\begin{array}{l} 4\notin\Sigma\ or \\ 4,7\in\Sigma, 8\notin\Sigma. \end{array}\right. \\ if & \{4,7,8\}\subseteq\Sigma & & then & e\geq 12 & (d-8+e\geq c) \\ 2d-6 & \underline{(0,6)}(2,4)(3,3) & \underline{(c,10)} & & & \\ 2d-7 & \underline{(0,7)}(2,5)(3,4) & \underline{(c,11)} & & & \end{array}\right.$$

If $t=3$, then $6\notin\Sigma$: we get $s_m=2d-7$.

If $t\geq 4$ and $4\in\Sigma$, we have $[d'-\ell,d']\cap\mathbb{N}\subseteq S$ and $\widetilde{s}\leq 2d'-d$. Then $s_m=\widetilde{s}+d\geq c+d-e$ by (3.11).

If $4\notin S$, by the above table we deduce that $s_m=2d-5$. $\diamond$

4.4 Semigroups with CM type $\tau \leq 7$.

As a consequence of the above results we obtain lower bounds or the exact value of s_m for semigroups with small Cohen-Maculay type. First, in the next lemma we collect well-known or easy relations among the CM type τ of S and the other invariants.

Lemma 4.8 *Let τ be the CM-type of the semigroup S as in* (2.1).

(1) $\#H+\ell\leq\tau\leq e-1$

(2) *Assume $\tau=\ell$, then $H=\emptyset$ and the following conditions are equivalent:*

- (a) $\ell=e-1$
- (b) $\tau=\ell,\ c'=d.$
- (c) $d=c-e.$
- (d) $S=\{0,e,2e,...,ke\to\}.$

(3) *If $c'>c-e$, then $\tau\geq\ell+1$ and $\tau=\ell+1\Longrightarrow H=\{c'-1\}$.*

(4) *Assume $\widetilde{s}\leq d'$ and $\tau=\ell+1$. Then* $\left[\begin{array}{ll} e\in\{2\ell+t-1,2\ell+t\}, & if\ \ \widetilde{s}=d' \\ e=2\ell+t, & if\ \ \widetilde{s}<d'. \end{array}\right.$

Proof. (1) Clearly every gap $h\geq c-e$ belongs to $S(1)\setminus S$, in particular $\{d+1,...,d+\ell\}\cup H\subseteq S(1)\setminus S$. The inequality $\tau\leq e-1$. is well-known.
(2) (3) are almost immediate.
(4). We have $\#H\leq 1$ by (1), $\widetilde{s}-\ell<\widetilde{s}\leq d'<c'-1$. If $\#H=0$, then $c'=c-e$ (4.1) and so $e=2\ell+t$ (2.7.1). If $\#H=1\Longrightarrow\widetilde{s}-\ell\notin H$ and $d'=c'-2$: it follows $e\leq 2\ell+t$ by (2.6.4). Now apply (2.6.3) and (2.8.2): if $\widetilde{s}<d'$, then $\widetilde{s}+1\in S$, hence $e\geq 2\ell+t$ and so $e=2\ell+t$. Further $\widetilde{s}=d'\Longrightarrow c'\geq c-e+\ell\Longrightarrow e\geq c-c'+\ell=d+2\ell-d'-1=2\ell+t-1$.

Example 4.9 (1) We recall that in general $c' = c - e$ does not imply $\tau = \ell$. For instance, let $S = \{0, 10_{e=d'}, 16_{c-e}, 17, 18, 19, 20, 21, 22, 23, 24_d, 26_c \rightarrow\}$. Then $\tau = 5 \neq \ell$.

(2) Analogously $H = \{c' - 1\}$ does not imply $\tau = \ell + 1$:
$S = \{0, 10_e, 16_{c-e}, 17, 18, 19, 20_{d'}, 22_{c'=d}, 26_c \rightarrow\}$ has $\ell = 3,\ \tau = 5$.

(3) In (4.8.4) the conditions $e = 2\ell + t$ and $\widetilde{s} < d'$ are not equivalent, further when $\widetilde{s} = d'$ both the cases with $\tau = \ell + 1,\ \widetilde{s} = d'$ are possible. For instance
$\{0, 9_{e=c-e}, 10, 11_{d'}, 13_{c'}, 14_d, 18_c \rightarrow\}$ has $t = \ell = 3,\ e = 2\ell + t,\ \widetilde{s} = d',\ \ \tau = \ell + 1$;
$\{0, 8_e, 9_{d'}, 11_{c'}, 12_d, 16_c \rightarrow\}$ has $t = \ell = 3,\ e = 2\ell + t - 1,\ \widetilde{s} = d',\ \ \tau = \ell + 1$.

(4) There exist semigroups with $H = \{c' - 1\},\ \widetilde{s} < d',\ \tau = \ell + 1$ as in (4.8.4):
$S = \{0 * ... * 11_e * * * 15_{d-e} * * * 19, 20, 21, 22, 23_{d'} * 25_{c'}\ 26_d * * * 30_c \rightarrow\}$,
has $\ell = 3,\ t = 5,\ e = 2\ell + t,\ \tau = 4$.

Now we deduce bounds for s_m when $\tau \leq 7$.

Proposition 4.10 *For each $\tau \leq 7$ we have $s_m \geq c + d - e$. More precisely when $\widetilde{s} < c' + d' - d$ we have the following results.*

(1) $\tau \leq 3$. We have: $s_m = \left[\begin{array}{ll} 2d - 4 & \text{if } S \text{ non-acute},\ \tau = t = 3\ (\ell = 2) \\ \widetilde{s} + d & \text{in the other cases} \end{array}\right.$
[6, 5.9] [5, 4.13]

(2) $\tau = 4$. We have $\ell \leq 4$ and the following subcases.

If $\ell = 4 (= \tau)$, then $H = \emptyset$ (4.8.1), therefore S is acute with $s_m = \widetilde{s} + d$ (2.3.4).
If $\ell \leq 3$ we are done by the previous (4.5), (4.7), (2.3.4) and (4.1) (recall $\ell = 1 \Longrightarrow S$ is acute).
More precisely we get:

$$s_m = \left[\begin{array}{lll} 2d - 4 & if & \left[\begin{array}{l} \ell = 2 \textit{ and either } (t = 3,\ d - 6 \notin S) \textit{ or } (t \geq 5,\ d - 3 \notin S) \\ \ell = t = 3, e = 9,\ c' = d,\ d' = c' - 2,\ d - 4 \in S \end{array}\right. \\ & & \\ 2d - 6 & if & \left[\begin{array}{l} \ell = 2,\ t = 3, d - 6 \in S \\ \ell = 3,\ t = 5,\ e = 11,\ c' = d - 1, d' = c' - 2 \end{array}\right. \\ & & \\ \widetilde{s} + d & \multicolumn{2}{l}{\textit{in the other cases.}} \end{array}\right.$$

(3) $\tau = 5$. As above we know s_m in every case:

- (a) If $\ell \leq 3$ we can deduce s_m by (4.5), (4.7),
- (b) If $4 \leq \ell \leq 5$, then we are done by (4.1), since $\#H \leq 1$.

(4) $\tau = 6$. We can calculate s_m as follows:

- (a) If $\ell \leq 3$ as in (3.a).
- (b) If $5 \leq \ell \leq 6$, then we are done by (4.1), since $\#H \leq 1$.
- (c) If $\ell = 4$ and $H \subseteq \{c' - 2, c' - 1\}$, we have the value of s_m by (4.1) and (4.1.2).
- (d) If $\ell = 4$ and $H = \{d' - k, c' - 1\}$, with $k \geq 1$, then $d' = c' - 2$, and the bounds for s_m are given in (3.11) if $k > \ell$, and in (4.4) if $\ell > k \geq 1$ (in fact $c' - d' = 2 \leq k + 1$).

(5) $\tau = 7$. We have $\ell \leq 7$ and the following subcases.

- (a) If $\ell \leq 3$, then s_m is known as in (3.a).
- (b) If $6 \leq \ell \leq 7$ then $\#H \leq 1$ and we are done by (4.1).
- (c) If $\ell = 5$, then $\#H \leq 2$ and we are done by (4.1), (4.4).

(d) If $\ell = 4$, then $\#H \leq 3$ and we are done if $\#H \leq 2$ by (4.1), (4.4).
If $\ell = 4$, $\#H = 3$, consider the following subcases

(i) $H = [d'+1, c'-1] \cap \mathbb{N}$: then s_m is given in (4.1).

(ii) $H = \{d'-k, c'-2, c'-1\}$, $k \geq 2$. If $k < \ell$, then s_m is given in (4.3.3). If $k \geq \ell$, apply (3.11).

(iii) $H = \{d'-1, c'-2, c'-1\}$: this case cannot exist. In fact since $S = \{e, ..., d'-2, *, d', *, *, c', ..., d, c \rightarrow\}$ and by the assumption $\widetilde{s} < d'$, we obtain $c' - \ell \in S$ and $c' - \ell = c' - 4 = d' - 1 \notin S$, impossible.

(iv) $H = \{d'-j, d'-k, c'-1\}$, $j > k \geq 1$, hence
$S = \{e, ..., d'-2, ..., d', *, c', ..., d(\leq c'+2), ..., d+5 \rightarrow\}$, with $2 = c' - d' \leq k+1$. As in the proof of (4.3.2) for $s := d' - k + c' - 1$ we have $\gamma(s) = -1$, $s + 1 - c' \notin S$, $s - d \in S \Longleftrightarrow s - d \neq d' - j$. Hence (Table 3.3 (b)) $s_m \geq s$ if $s - d \neq d' - j$, i.e., $d' - k + c' - 1 - d \neq d' - j$, i.e., $d - c' \neq j - k - 1$.

$$\text{Four subcases:} \left[\begin{array}{lcl} j = k+1 & \Longrightarrow & s_m \geq s \;\; if \;\; d \neq c' \\ j = k+2 & \Longrightarrow & s_m \geq s \;\; if \;\; d \neq c'+1 \\ j = k+3 & \Longrightarrow & s_m \geq s \;\; if \;\; d \neq c'+2 \\ j \geq k+4 & \Longrightarrow & s_m \geq s \\ & \text{(since} & d - c' \leq \ell - 2 = 2, \; j - k - 1 \geq 3). \end{array} \right.$$

In the remaining three situations we can see that $[d' - \ell, d'] \cap \mathbb{N} \subseteq S$, therefore s_m is given by (3.11):
- If $j = k+1$ and $d = c'$, since $\widetilde{s} < d'$, $\ell = 4$ we get $\{d'-4, d-4 = d'-2, d'\} \subseteq S$. Since there are two consecutive holes, then $k \geq 5$. It follows $[d' - \ell, d'] \cap \mathbb{N} \subseteq S$.
- If $j = k+2$, and $d = c'+1$, we have $c' - 4 = d' - 2$, $d' - 1 = d - 4$ and $\widetilde{s} < d' - 1$ (4.3.1) . Therefore $S = \{e, ..., d'-5, d'-4, d'-3, *, d'-2, d'-1, d', *, d'+2 = c', d'+3 = d, d+5 \rightarrow\}$. We deduce $[d' - \ell, d'] \cap \mathbb{N} \subseteq S$.
- If $j = k+3$, and $d = c'+2$, analogously we deduce $[d' - \ell, d'] \cap \mathbb{N} \subseteq S$. $\diamond$

4.5 The value of s_m for semigroups of multipicity $e \leq 8$.

Corollary 4.11 *For each semigroup S of multiplicity $e \leq 8$ we have $s_m \geq c + d - e$.*

Proof. Since $\tau \leq e - 1$ the result follows by (4.10). $\diamond$

4.6 Almost arithmetic sequences and Suzuki curves.

Recall that a semigroup S is *generated by an almost arithmetic sequence* (shortly AAS) if

$$S =< m_0, m_1, ..., m_{p+1} \; , n >$$

with $m_0 \geq 2$, $m_i = m_0 + \rho\, i$, $\forall\; i = 1, ..., p+1$, *and* $GCD(\rho, m_0, n) = 1$. (The embedding dimension of S is *embdim* $S = p + 2$).

Proposition 4.12 *Let S be an AAS semigroup of embedding dimension μ; then $\tau \leq 2(\mu - 2)$.*

Proof. It is a consequence of [8, 3.3 - 4.6 - 4.7 - 5.6 - 5.7 - 5.8 - 5.9] after suitable calculations.

Corollary 4.13 *If S is an AAS semigroup with embdim $S \leq 5$ then $s_m \geq c + d - e$.*

Proof. It is an immediate consequence of (4.10) and (4.12). $\diamond$

As another corollary we obtain the value of s_m for the Weierstrass semigroup of a *Suzuki curve*, that is a plane curve C defined by the equation

$$y^b - y = x^a(x^b - x), \text{ with } a = 2^n, \; b = 2^{2n+1}, \; n > 0.$$

Some applications of these curves to AG codes can be found for example in [4].

Proposition 4.14 *If S is the Weierstrass semigroup of a Suzuki Curve, then S is symmetric, therefore $s_m = \widetilde{s} + d$.*

Proof. In [4, Lemma 3.1] is proved that the Weierstrass semigroup S at a rational place of the function field of C is generated as follows:

$$S =< b, b+a, b+\frac{b}{a}, 1+b+\frac{b}{a} >$$

We have $b = 2a^2$, with $a = 2^n$, hence $S =< 2a^2,\ 2a^2+a,\ 2a^2+2a,\ 2a^2+2a+1 >$ Then consider the semigroup

$$S =< 2a^2,\ 2a^2+a,\ 2a^2+2a,\ 2a^2+2a+1 >, \quad a \in \mathbb{N}.$$

If $a = 1$, then $S =< 2, 3 >$.
If $a > 1$, then S is generated by an almost arithmetic sequence, and $embdim(S) = 4$; in fact

$$S =< m_0, m_1, m_2, n >, \quad \text{with } m_0 = 2a^2,\ m_1 = m_0 + a,\ m_2 = m_1 + a,\ n = m_2 + 1.$$

Since S is AAS, we shall compute the Apery set $\mathcal{A}$ by means of the algorithm described in [8]:
let $p = embdim(S) - 2 = 2$ and for each $t \in \mathbb{N}$,
let q_t, r_t be the (uniqe) integers such that $t = pq_t + r_t (= 2q_t + r_t)$, $q_t \in \mathbb{Z}$, $r_t \in \{1, 2\}$,
let $g_t = q_t m_2 + m_{r_t}$, i.e., $g_t = \left[\begin{array}{ll} (q_t+1)m_2 & if\ r_t = 2 \\ q_t m_2 + m_1 & if\ r_t = 1 \end{array}\right.$ (in particular $g_0 = 0$).
Then by [8] the Apery set $\mathcal{A}$ of S is: $\{\ g_t + hn \ \mid\ 0 \le t \le 2a-1,\ 0 \le h \le a-1\ \}$: therefore the elements of the Apery set are the $2a^2$ entries of the following matrix

$$\left[\begin{array}{cccccc} 0 & g_1 & g_2 & g_3 & \cdots & g_{2a-1} \\ & \| & \| & \| & \cdots & \| \\ & m_1 & m_2 & m_1+m_2 & \cdots & (a-1)m_2+m_1 \\ n & g_1+n & g_2+n & . & \cdots & . \\ 2n & g_1+2n & g_2+2n & . & \cdots & . \\ . & . & . & . & \cdots & . \\ . & . & . & . & \cdots & . \\ (a-1)n & g_1+(a-1)n & g_2+(a-1)n & . & \cdots & g_{2a-1}+(a-1)n \end{array}\right]$$

Recall that a semigroup S of multiplicity e and Apery set $\mathcal{A}$ is
symmetric $\Longleftrightarrow$ for each $s_i \in \mathcal{A}$, $0 < s_i \neq s_e := max\,\mathcal{A}$, there exists $s_j \in \mathcal{A}$ such that $s_i + s_j = s_e$.
In our case c this condition is satisfied: in fact $s_i = \left[\begin{array}{lll} \alpha m_2 + h\,n, & h \le a-1,\ \alpha \ge 1 & (1)\ or \\ \alpha\, m_2 + m_1 + h\,n & 0 \le \alpha,\ h \le a-1 & (2) \end{array}\right.$
further $s_e = (a-1)(m_2+n) + m_1$, and so

$$s_e - s_i = \left[\begin{array}{ll} (a-1-\alpha)m_2 + m_1 + (a-1-h)\,n \in \mathcal{A} & (1)\ or \\ (a-1-\alpha)\,m_2 + (a-1-h)\,n \in \mathcal{A} & (2) \end{array}\right.$$

Since a semigroup S is symmetric if and only if its CM-type is one, then $s_m = \widetilde{s} + d$ by (4.10.1). $\diamond$

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