# A classification of one-dimensional local domains based on the invariant $(c-\delta)r-\delta$.

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**Abstract**. Let $(R,\mathfrak{m})$ be a one-dimensional, local, Noetherian domain and let $\overline{R}$ be the integral closure of $R$ in its quotient field $K$. We assume that $R$ is not regular, analitycally irreducible and residually rational. The usual valuation $v : K \longrightarrow \mathbb{Z} \cup \infty$ associated to $\overline{R}$ defines the numerical semigroup $v(R) = \{v(a),\ a \in R,\ a \neq 0\} \subseteq \mathbb{N}$. The aim of the paper is to study the non-negative invariant $b := (c-\delta)r-\delta$, where $c, \delta, r$ denote the conductor, the length of $\overline{R}/R$ and the Cohen Macaulay type of $R$, respectively. In particular, the classification of the semigroups $v(R)$ for rings having $b \leq 2(r-1)$ is realized. This method of classification might be successfully utilized with similar arguments but more boring computations in the cases $b \leq q(r-1)$, for reasonably low values of $q$. The main tools are type sequences and the invariant $k$ which estimates the number of elements in $v(R)$ belonging to the interval $[c-e, c)$, $e$ being the multiplicity of $R$.

**Introduction.** Let $(R,\mathfrak{m})$ be a one-dimensional, local, Noetherian domain and let $\overline{R}$ be the integral closure of $R$ in its quotient field $K$. We assume that $R$ is not regular and analitycally irreducible, i.e. $\overline{R}$ is a DVR with uniformizing parameter $t$ and a finite $R$-module. We also suppose $R$ to be residually rational, i.e. $R/\mathfrak{m} \simeq \overline{R}/t\overline{R}$. Called $v : K \longrightarrow \mathbb{Z} \cup \infty$ the usual valuation associated to $\overline{R}$, the image $v(R) = \{v(a),\ a \in R,\ a \neq 0\} \subseteq \mathbb{N}$ is a numerical semigroup. Starting from the following classical invariants:

$c$, the *conductor* of $R$, i.e. the minimal $j \in v(R)$ such that $j + \mathbb{N} \subset v(R)$,

$\delta := \ell_R(\overline{R}/R)$, the number of gaps of the semigroup $v(R)$ in $\mathbb{N}$,

$r := \ell_R((R : \mathfrak{m})/R)$, the *Cohen Macaulay type* of $R$,

the new invariant

$$b := (c-\delta)r-\delta$$

has been recently considered in the literature. The general problem of classifying rings according to the size of $b$ has been examined by several authors. First, Brown and Herzog in [2] characterize all the one-dimensional reduced local rings having $b = 0$ or $b = 1$. Successively, in [3], [4], [6], Delfino, D'Anna and Micale

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consider the rings for which $b \le r$. Partial answers in the case $b > r-1$ are given in [5].

In [10, Section 4] we obtain some improvements of the quoted results. This is done by using the expression of the invariant $b$ in terms of the type sequence $[r_1, .., r_n]$ (defined in (1.1)), where $n := c-\delta$ and $r_1$ equals the Cohen-Macaulay type $r$ of $R$, namely:

$$b = \sum_{i=1}^{n}(r - r_i).$$

So, employing the properties of the type sequence, we get as a straightforward consequence of the preceding formula the well known bounds

$$0 \le b \le (n-1)(r-1)$$

(for the positivity see [2], Theorem 1; for the upper bound see [3], Proposition 2.1). Also, we recover in an immediate way the two extremal cases:

$b = 0$, corresponding to the so called rings *of maximal length*, i.e. the rings having maximal type sequence $[r, r, ..., r]$;

$b = (n-1)(r-1)$, corresponding to the *almost Gorenstein* rings, i.e. the rings having minimal type sequence $[r, 1, ..., 1]$.

Actually, for any integer $q \in \mathbb{N}$ it is natural to ask if it is possible to characterize the rings verifying

$$(q-1)(r-1) \le b \le q(r-1).$$

In Section 3 we write explicitly all the possible values of $v(R)$ for $1 \le q \le 2$ (see Theorems (3.3), (3.4), (3.6)), but we outline that the method used here is absolutely general and analogous although more tedious computations might be repeated for greater values of $q$.

To achieve our results, we utilize heavily the number

$$k := \ell_R(R/(\mathfrak{C} + xR)),$$

where $\mathfrak{C} := t^c\overline{R}$ denotes the *conductor ideal* of $R$ in $\overline{R}$ and $x$ an element of $R$ such that $v(x) = e(R)$, the *multiplicity.* In [5] it is established that $b = r-1 \Longrightarrow k = 1$ or 2 [5, Proposition 2.4], and that $b = r-1$ and $k = 2 \Longrightarrow r = e-2$ [5, Corollary 2.13]. In [6] the lower bound $rk - e + 1 \le b$ is found. Improvements of these results and several other inequalities relating the invariants $k, b, r$ are now realized by means of the type sequence of $R$ (see (1.4) and (2.1)). For this purpose we introduce in Section 1 a decomposition of the semigroup $v(R)$ as a disjoint union of subsets:

$$v(R) = \{0, e, 2e, ..., pe, c, \rightarrow\} \cup H_1 \cup .... \cup H_{k-1},$$

where $H_i := \{y_i, y_i+e, ..., y_i+l_ie\}$, $i = 1, ..., k-1$, $p$, $l_i \in \mathbb{N}$, and $\{y_i\}_{i=1,..,k-1}$ have distinct residues (mod $e$) (see Setting 1.6). This allows us to obtain in Section 2 the useful formula (2.2.1):

$$b = X + Y + Z$$

where $X := (k-1)(r-1) \ge 0$,

$Y := k - (e-r) \ge 0$,

$Z := (r+1)(p + \sum_1^{k-1} l_i) + k + h - pe - 1 \ge 0.$

Obviously $X + Y = rk - e + 1$, and so the integer $Z$ measures how far is $b$ from the lower bound proved in [6].

The advantage of this formula is evident for low values of $b$. For instance, for rings having $b \in \{0, 1, 2\}$ we state in a quite simple way all the possible value sets (see Theorems (3.1), (3.8), (3.9)). Nevertheless, a such type of classification might be accomplished for greater values of $b$ with similar arguments.

# 1 Preliminary results.

We begin by giving the setting of the paper.

**Setting 1.1** *(set2)* Let $(R, \mathfrak{m})$ be a one-dimensional local Noetherian domain with residue field $k$ and quotient field $K$. We assume throughout that $R$ is not regular with normalization $\overline{R} \subset K$ a DVR and a finite $R$-module, i.e., $R$ is analytically irreducible. Let $t \in \overline{R}$ be a uniformizing parameter for $\overline{R}$, so that $t\overline{R}$ is the maximal ideal of $\overline{R}$. We also suppose that the field $k$ is isomorphic to the residue field $\overline{R}/t\overline{R}$, i.e., $R$ is residually rational. We denote the usual valuation on $K$ associated to $\overline{R}$ by $v$; that is, $v : K \longrightarrow \mathbb{Z} \cup \infty$, and $v(t) = 1$. By [9, Proposition 1] in this setting it is possible to compute for a pair of fractional nonzero ideals $I \supseteq J$ the length of the $R$-module $I/J$ by means of valuations:

$$\ell_R(I/J) = |v(I) \setminus v(J)|. \tag{1.1.1}$$

The set $v(R) := \{v(a) \mid a \in R,\ a \neq 0\} \subseteq \mathbb{N}$ is the *numerical semigroup* of $R$. Since the *conductor* $\mathfrak{C} := (R :_K \overline{R})$ is an ideal of both $R$ and $\overline{R}$, there exists a positive integer $c$ so that $\mathfrak{C} = t^c\overline{R}$, $\ell_R(\overline{R}/\mathfrak{C}) = c$ and $c \in v(R)$. Furthermore, denoting by $\delta := \ell_R(\overline{R}/R)$ the number of gaps of the semigroup $v(R)$ and $r := \ell_R((R : \mathfrak{m})/R)$ the *Cohen Macaulay type* of $R$, we define the invariant

$$b := (c - \delta)r - \delta.$$

We list the elements of $v(R)$ in order of size: $v(R) := \{s_i\}_{i \geq 0}$, where $s_0 = 0$ and $s_i < s_{i+1}$, for every $i \geq 0$. We put $e := s_1$ the *multiplicity* of $R$ and $n = c - \delta$ the number such that $s_n = c$. For every $i \geq 0$, let $R_i$ denote the ideal of elements whose values are bounded by $s_i$, that is,

$$R_i := \{a \in R \mid v(a) \geq s_i\}.$$

The ideals $R_i$ give a strictly decreasing sequence

$$R = R_0 \supset R_1 = \mathfrak{m} \supset R_2 \supset \ldots \supset R_n = \mathfrak{C} \supset R_{n+1} \supset \ldots ,$$

which induces the chain of duals:

$$R \subset (R : R_1) \subset \ldots \subset (R : R_n) = \overline{R} \subset (R : R_{n+1}) = t^{-1}\overline{R} \subset \ldots .$$

Put $r_i := l_R((R : R_i)/(R : R_{i-1}))$, $i \geq 1$; the finite sequence of integers $[r_1, \ldots, r_n]$ is the *type sequence* of $R$.

In particular $r_1 = r$, the Cohen-Macaulay type of $R$. Moreover it is known that:

- $1 \leq r_i \leq r$ for every $i \geq 1$, and $r_i = 1$ for every $i > n$,
- $\delta = \sum_1^n r_i$,
- $2\delta - c = \sum_1^n (r_i - 1) = \sum_1^\infty (r_i - 1)$ (see, e.g. [10, Prop.2.7]).

Type sequence is a suitable tool to study the behavior of the invariant $b$.

**Proposition 1.2** *We have:*

(1) $b = \sum_{i=1}^n (r - r_i)$.

(2) $0 \leq b \leq (n - 1)(r - 1)$.

Proof. For (1) see [10, Section 4].
(2). We have: $\sum_{i=1}^{n}(r-r_i)=\sum_{i=2}^{n}(r-r_i)\leq(n-1)(r-1)$, because $r_1=r$ and $r_i\geq 1$, for every $i\geq 1$. $\diamond$

**Notation 1.3** Let $R$ be as in (1.1). We set:

- $x\in\mathfrak{m}$ is an element such that $v(x)=e$; namely, $\ell_R(R/xR)=e$.
- For $a,b\in\mathbb{Z}$, $[a,b]=\{x\in\mathbb{Z}\mid a\leq x\leq b\}$.
- $i_0\in[1,n]$ is such that $s_{i_0-1}=min\{y\in v(R)\mid y\geq c-e\}$.
  $(i_0=1\iff c=e)$.
- $B:=[i_0,n]$ and $A:=[1,n]\setminus B$ $(|A|\leq n-1)$.
- $k:=\ell_R(R/(\mathfrak{C}+xR))$ $(1\leq k\leq e-1)$.

**Theorem 1.4** *The following facts hold.*

(1) $k=|B|=\ell_R(\mathfrak{C}:_R\mathfrak{m}/\mathfrak{C})\geq e-r>0$.

(2) $k\leq\sum_{i\in B}r_i\leq e-1$. *If* $\sum_{i\in B}r_i=e-1$, *then* $s_{i_0-1}=c-e$.

Proof. (1) and the inequality $\sum_{i\in B}r_i\leq e-1$ of (2) are proved in [10, Lemma 4.2]. Since $r_i\geq 1$ for every $i$ and $|B|=k$, the inequality $k\leq\sum_{i\in B}r_i$ is done.
Moreover, denoting by $\omega$ the canonical module of $R$ (see [10] for the existence and the properties in our setting), we remark that
$\sum_{i\in B}r_i=\ell_R(\overline{R}/(R:R_{i_0-1}))=|v(\omega R_{i_0-1})_{<c}|$ and
$v(\omega R_{i_0-1})_{<c}\subseteq[c-e,c-2]$
(see the proof of the quoted lemma). Thus $\sum_{i\in B}r_i=e-1\Longrightarrow v(\omega R_{i_0-1})_{<c}=[c-e,c-2]$, and so $s_{i_0-1}$, the minimal element in $v(\omega R_{i_0-1})$, equals $c-e$. $\diamond$

The case $k=1$ is completely known and recalled below for the convenience of the reader.

**Proposition 1.5** [10, Lemma 4.4] *The following facts are equivalent:*

(1) $k=1$.

(2) $v(R)=\{0,e,...,pe,c\rightarrow\}$.

(3) *The type sequence of* $R$ *equals* $[e-1,....,e-1,r_n]$.

*If* $R$ *satisfies these equivalent conditions, then:*

$$\delta=c-p-1,\ b=(p+1)e-c\leq r-1,\ r=e-1,\ r_n=e-1-b.$$

By virtue of (1.1.1) we have $k=|v(R)\setminus v(\mathfrak{C}+xR)|$. This fact allows to write $v(R)=v(\mathfrak{C}+xR)\cup\{0,y_1,...,y_{k-1}\}$, obtaining the description of $v(R)$ as a disjoint union of the sets $H_i$ given in the next setting. The construction is significant for $k>1$.

**Setting 1.6** Let $k > 1$. We set:

$v(R)= \{0, e, 2e, ..., pe, c, \rightarrow\} \cup H_1 \cup .... \cup H_{k-1}$, where

- $p$ is the integer such that $c-e \leq pe < c$, in other words, $pe+2 \leq c \leq (p+1)e$. ($p \geq 0$ and $p = 0 \iff c = e$).
- $h := (p+1)e - c, \quad (0 \leq h \leq e-2)$.
- $H_i := \{y_i, y_i + e, ..., y_i + l_i e\},\ i = 1, ..., k-1,\ l_i \in \mathbb{N}$.
- The integers $y_i \in \mathbb{N}$ are such that $e < y_1 < y_2 < ... < y_{k-1}, \quad y_i \notin e\mathbb{Z}$, $\overline{y_i} \neq \overline{y_j}\ (mod\ e)$ for every $i, j \in \{1, .., k-1\}$.
- The integers $l_i,\ i = 1, ..., k-1$, are defined by the relations:
  $y_i + l_i e < c \leq y_i + (l_i + 1)e$.
- For $k = 2$ we shortly call $y := y_1, \quad l := l_1$.

**Example 1.7** If $v(R) =< 10, 11, 26 >$, then:
$v(R) = \{0, 10, 20, 30, 40, 50 \rightarrow\} \cup H_1 \cup ... \cup H_7$ where $H_1 = \{11, 21, 31, 41\},\ H_2 = \{22, 32, 42\},\ H_3 = \{26, 36, 46\},\ H_4 = \{33, 43\},\ H_5 = \{37, 47\},\ H_6 = \{44\},\ H_7 = \{48\}$. According to notations previously introduced $y_1 = 11, y_2 = 22, y_3 = 26, y_4 = 33, y_5 = 37, y_6 = 44, y_7 = 48$ and $l_1 = 3, l_2 = l_3 = 2, l_4 = l_5 = 1, l_6 = l_7 = 0$. Moreover, $c = 50, p = 4, h = 0$.

**Proposition 1.8** *Let* $k, k > 1, p, h, \{l_i\}$ *be the integers defined in* (1.3) *and* (1.6). *Then:*

(1) $r \in \{e-k, ...., e-1\}$.

(2) $0 \leq l_{k-1} \leq .... \leq l_2 \leq l_1 \leq p-1$.

(3) $c - \delta = p + k + \sum_1^{k-1} l_i$,
$\delta = (p+1)(e-1) - h - \sum_1^{k-1}(l_i + 1)$.

Proof. Assertion (1) follows immediately from (1.4.1).
(2). By definition of $l_i$ and $p$, we have $(l_i + 1)e < y_i + l_i e < c \leq (p+1)e$; then $l_i + 1 \leq p$, for every $i = 1, ..., k-1$. Now note that

$$y_i + l_i e < c \leq y_{i-1} + (l_{i-1}+1)e \Longrightarrow y_i - y_{i-1} < (l_{i-1} + 1 - l_i)e \Longrightarrow l_i \leq l_{i-1}.$$

(3). Using the integers defined in (1.6) $c - \delta$ and $\delta$ can be expressed as :

$$c - \delta = (p+1) + (l_1 + 1) + ... + (l_{k-1} + 1) = p + k + \sum_1^{k-1} l_i,$$
$$\delta = c - (c - \delta) = (p+1)e - h - (p + k + \sum_1^{k-1} l_i)$$
$$= (p+1)(e-1) - h - \sum_1^{k-1}(l_i + 1). \quad \diamond$$

It is natural to ask how the elements $y_1, ..., y_{k-1}$ introduced in (1.6) influence the Cohen Macaulay type of $R$. This will be analysed in the following (1.9), (1.11), (1.12).

**Proposition 1.9** *Let $k = \ell_R(R/(\mathfrak{C}+xR))$ and let $v(R)$ be as in* (1.6)*. Further let $s_1, ..., s_{k-1} \in \mathfrak{m}$ be such that $v(s_i) = y_i$. The following facts are equivalent:*

(1) $r = e - 1$.

(2) $v(R) \setminus v(xR : \mathfrak{m}) = \{0\}$.

(3) $y_1, ..., y_{k-1} \in v(xR : \mathfrak{m})$.

(4) $s_1, ..., s_{k-1} \in (xR : \mathfrak{m})$.

(5) $s_i s_j \in x\mathfrak{m}$ *for every* $i, j = 1, ..., k-1$.

Proof. Since $e - r = \ell_R(R/xR) - \ell_R((xR : \mathfrak{m})/xR) = \ell_R(R/(xR : \mathfrak{m}))$, the equality $e - r = 1$ means $|v(R) \setminus v(xR : \mathfrak{m})| = 1$, and so $1 \iff 2$ is proved. In the same way we obtain that
$(*)$ $r = e - 1 \iff (xR : \mathfrak{m}) = \mathfrak{m} \iff \mathfrak{m}^2 = x\mathfrak{m}$. Moreover,
$(**)$ $v(x^{-1}\mathfrak{m}) \subseteq \mathbb{N} \Longrightarrow x^{-1}\mathfrak{m}\mathfrak{C} \subseteq \mathfrak{C} \Longrightarrow \mathfrak{m}\mathfrak{C} = x\mathfrak{C} \Longrightarrow \mathfrak{C} \subseteq (xR : \mathfrak{m})$.
Considering the chain of ideals
$R \supset \mathfrak{m} \supseteq \mathfrak{C} + (x, s_1, ..., s_{k-1})R \supset \mathfrak{C} + (x, s_1, ..., s_{k-2})R \supset ... \supset \mathfrak{C} + xR$,
we see that $\ell_R(R/(\mathfrak{C} + xR)) = k \Longrightarrow \mathfrak{m} = \mathfrak{C} + (x, s_1, ..., s_{k-1})R$, hence
$(***)$ $s_i\mathfrak{m} = (xs_i)R + (s_i s_j)R + s_i\mathfrak{C}$ for every $j = 1, ..., k-1$.
By $(*)$ we have immediately $1 \Longrightarrow 5$.
$5 \Longrightarrow 4$. By the assumption $s_i s_j \in x\mathfrak{m}$, $\forall\, i, j = 1, ..., k-1$ and by the obvious inclusion $s_i\mathfrak{C} \subseteq \mathfrak{m}\mathfrak{C} = x\mathfrak{C}$, from $(***)$ we get $s_i\mathfrak{m} \subseteq xR$, then $s_i \in (xR : \mathfrak{m})$.
The implication $4 \Longrightarrow 3$ is obvious.
Finally, $3 \Longrightarrow 2$ holds by $(**)$. $\diamond$

**Remark 1.10** It is clear from (1.9) (see the equivalence 1-5) that the condition $y_i + y_j - e \in v(R)$ for every $i, j = 1, ..., k-1$, is necessary to have maximal Cohen Macaulay type. Unfortunately, it is not sufficient. For example, if $R = k[[t^6, t^9 + t^{10}, t^{14}, t^{16}, t^{17}, t^{19}]]$, then $k = 2$, $y = 9$, $2y - e = 12 \in v(R)$, but $r = e - 2$. In this case (1.9.5) does not hold, because $(t^9 + t^{10})^2 \notin xR$.

**Proposition 1.11** *Let $k = \ell_R(R/(\mathfrak{C} + xR))$ and let $v(R)$ be as in* (1.6)*.*

(1) $r = e - k,\ k \geq 2, \iff v(R) \setminus v(xR : \mathfrak{m}) = \{0, y_1, ...., y_{k-1}\}$.

(2) *If $r < e - 1$, then*
  (a) $2y_1 < c + e$,
  (b) $p \leq 2l_1 + 2$ *and* $p = 2l_1 + 2 \Longrightarrow h > 0$.

(3) *If $r = e - k$, then*
  (a) $y_1 + y_j < c + e$, *for every* $j = 1, ..., k-1$.
  (b) $p \leq l_1 + l_{k-1} + 2$ *and* $p = l_1 + l_{k-1} + 2 \Longrightarrow h > 0$.

(4) *If $p \geq 3$ and $i$ is such that $l_i = 0$, then* $2y_i > c + e$.

Proof. (1). By means of (∗∗) stated in the proof of (1.9), we have the inclusions $(\mathfrak{C} + xR) \subseteq (xR : \mathfrak{m}) \subseteq R$. Since $e - r = \ell_R(R/(xR : \mathfrak{m}))$ and $k = \ell_R(R/(\mathfrak{C} + xR))$, it follows that $e - r = k \iff (\mathfrak{C} + xR) = (xR : \mathfrak{m})$.
To see (2.a), suppose $2y_1 \geq c+e$, then $y_i + y_j \geq c+e$ for every $i, j = 1, ..., k-1$. Let $x_i \in \mathfrak{m}$ be elements such that $v(x_i) = y_i$ and let $s \in \mathfrak{m}$. If $s \in (\mathfrak{C}+xR)$, then $x_i s \in \mathfrak{m}(\mathfrak{C}+xR) \subseteq xR$. If $s \notin (\mathfrak{C}+xR)$, then $v(s) = y_j$, for some $j$, $1 \leq j \leq k-1$, hence $v(x_i s) = y_i + y_j \geq c+e \Longrightarrow x_i s \in x\mathfrak{C} \subset xR$. In both cases $x_i \in (xR : \mathfrak{m})$, and so $y_i \in v(xR : \mathfrak{m})$. Thus $v(R) \setminus v(xR : \mathfrak{m}) = \{0\}$ and $r = e - 1$ by (1.9), a contradiction.
To see (2.b), consider that by (1.3) and (1.6):
$y_1 \geq c - (l_1 + 1)e = (p - l_1)e - h.$
Combining this with the preceding (2.a), we obtain
$(2p - 2l_1)e - 2h \leq 2y_1 < c + e = (p + 2)e - h.$
Thus $(p - 2l_1 - 2)e < h$ and since $h \leq e - 2$, we see that $p \leq 2l_1 + 2$ and also that $p = 2l_1 + 2 \Longrightarrow h > 0$.
To prove (3.a), it suffices to show that $y_1+y_{k-1} < c+e$. Suppose $y_1+y_{k-1} \geq c+e$, then $y_i + y_{k-1} \geq c + e$ for all $i$. Let $x_{k-1} \in \mathfrak{m}$ be an element such that $v(x_{k-1}) = y_{k-1}$. As in (2.a), we get $x_{k-1} \in (xR : \mathfrak{m})$, and so $y_{k-1} \in v(xR : \mathfrak{m})$, a contradiction, since the assumption $e - r = k$ means $v(R) \setminus v(xR : \mathfrak{m}) = \{0, y_1, ...., y_{k-1}\}$ (see item 1).
We prove now (3b). As in (2.b),
$y_j \geq c - (l_j + 1)e = (p - l_j)e - h,$ for $j = 1, ..., k - 1$,
and by (3.a)
$(2p - l_j - l_1)e - 2h \leq y_1 + y_j < c + e = (p + 2)e - h$. Hence
$(p - l_j - l_1 - 2)e < h \leq e - 2$, for every $j = 1, ..., k - 1$.
We conclude
$p \leq l_1 + l_j + 2 \leq l_1 + l_{k-1} + 2$ and also the last assertion.
For (4), note that $l_i = 0 \Longrightarrow y_i + e \geq c$, and that $p \geq 3 \Longrightarrow c > 3e$.
Thus: $2y_i \geq 2c - 2e = c + (c - 2e) > c + e$, as desired. ⋄

We may describe the particular case $k = 2$ in a more precise way.

**Proposition 1.12** *Assume $k = 2$. With setting (1.6) we have:*

(1) $r = e - 1 \iff$ *one of the following conditions is satisfied:*

- (*a*) $2y \geq c + e$;
- (*b*) $2y = (2q + 1)e < c + e,\ q \geq 1,\ p \geq 2$ *and* $y \in v(xR : \mathfrak{m})$.

(2) $r = e - 2 \iff 2y < c + e$ *and if* $2y = (2q + 1)e$, *then* $y \notin v(xR : \mathfrak{m})$.

Proof. First recall that by (1.4.1) one has $r \geq e-2$. For implication $\Longrightarrow$ in (1), note that $y \in v(xR : \mathfrak{m})$, by (1.9), and so $2y - e \in v(\mathfrak{m})$. Then regarding the structure of $v(R)$, we have the claim. For the opposite implication, note that in case (*a*) for any $s \in \mathfrak{m}$ such that $v(s) = y$, $v(x^{-1}s^2) = 2y - e \geq c \Longrightarrow x^{-1}s^2 \in \mathfrak{C} \Longrightarrow s^2 \in x\mathfrak{m}$; now use again (1.9) to conclude.
(2) is immediate by (1). ⋄

## 2 Bounds for the invariant $b$.

Starting from the preliminary result (1.2) we go on in studying the integer $b$. First (see (2.1)) we find lower and upper bounds using the properties of the type sequence, then (see (2.2)) we express $b$ in terms of the integers $k, p, l_i, h$ occurring in the decomposition of $v(R)$ as in (1.6). This description becomes quite simple in the particular cases $k = 2, 3$ (see (2.3) and (2.4)). The last result of the present section (see(2.5)) furnishes informations according to the range $(q-1)(r-1) < b \leq q(r-1)$, that will be basic in the next section.

**Proposition 2.1** *With Notation* 1.3*, the following facts hold.*

(1) $(e-r-1)(r-1) \leq rk - e + 1 \leq b - \sum_{i \in A}(r - r_i) \leq k(r-1)$.

(2) $b = (k-1)(r-1) + \sum_{i \in A}(r - r_i) \iff \sum_{i \in B} r_i = e - 1$ *and* $k = e - r$.

(3) $b = k(r-1) + \sum_{i \in A}(r - r_i) \iff r_i = 1$ *for every* $i \in B$.

(4) *The following conditions are equivalent:*

- ($a$) $b = (e-r-1)(r-1)$.
- ($b$) $b = (k-1)(r-1)$.
- ($c$) $e - r = k$, $\sum_{i \in B} r_i = e - 1$ *and* $r_i = r$ *for every* $i \in A$.

*If these conditions hold, then* $s_{i_0 - 1} = c - e$.

(5) $b \geq (r-1)s$, *where* $s := |\{i \in [1, n] \mid r_i = 1\}|$.

Proof. Write the invariant $b = \sum_{i=1}^{n}(r - r_i)$ in the following form:

$$(*) \qquad b = \sum_{i \in B}(r - r_i) + \sum_{i \in A}(r - r_i) = rk - \sum_{i \in B} r_i + \sum_{i \in A}(r - r_i).$$

Using that $\sum_{i \in B} r_i \leq e - 1$ (see 1.4.2), we obtain

$$(**) \qquad rk - (e-1) \leq b - \sum_{i \in A}(r - r_i) \leq k(r-1).$$

Then, since $k \geq e - r$ by (1.4.1), the inequalities of (1) are clear.
(2). Supposing $b - \sum_{i \in A}(r - r_i) = (k-1)(r-1)$ we have by item 1 $(k-1)(r-1) \geq rk - e + 1$, hence $k \leq e - r$ and since always $k \geq e - r$, it follows that $k = e - r$. From $(*)$ $\sum_{i \in B} r_i = rk - (k-1)(r-1) = k + r - 1 = e - 1$. For the converse, it suffices to substitute $\sum_{i \in B} r_i = k + r - 1$ in $(*)$.
(3). Using $(*)$ we have $b - \sum_{i \in A}(r - r_i) = k(r-1) \iff \sum_{i \in B}(r - r_i) = k(r-1)$. Since $r - r_i \leq r - 1$ for every $i$ and $k = |B|$, the last fact is equivalent to say that $r_i = 1$ for every $i \in B$.
(4), $a \Longrightarrow b$. By (1) we have immediately $\sum_{i \in A}(r - r_i) = 0$ and $(e-r-1)(r-1) = rk - e + 1 \Longrightarrow e - r = k$, as desired.
$b \Longrightarrow c$. By (1) we have $\sum_{i \in A}(r - r_i) \leq b - (rk - e + 1) = -k - r + e \leq 0$, then we can apply item 2 with $\sum_{i \in A}(r - r_i) = 0$.
$c \Longrightarrow a$. Substitute in $(*)$ the relations of (c).
The fact $s_{i_0 - 1} = c - e$ is immediate by (1.4.2).

By applying [10, Corollary 3.13.2], with $I = \mathfrak{C}$ we get (5). $\diamond$

Utilizing the description of the value set $v(R)$ introduced in (1.6), we obtain next useful formula for the invariant $b$.

**Theorem 2.2** *With Setting 1.6, assume $k > 1$. The following equalities hold:*

(1) $b = (r+1)\sum_1^{k-1}(l_i+1) - (p+1)(e-r-1) + h = X + Y + Z$

*where* $X := (k-1)(r-1) \geq 0$,

$Y := k - (e-r) \geq 0$,

$Z := (r+1)(p + \sum_1^{k-1} l_i) + k + h - pe - 1 \ \geq \sum_{i\in A}(r - r_i) \geq 0.$

(2) $c = (p + 1 + \sum_1^{k-1}(l_i+1))(r+1) - b.$

Proof. (1). To get the desired formula it suffices to substitute in the equality $b = (c-\delta)r - \delta$ the expressions of $c-\delta$ and $\delta$ given in (1.8.3). The positivity of $Y$ is clear by (1.4.1). To prove the positivity of $Z$ we use the second inequality of (2.1.1): $X+Y = kr - e + 1 \leq b - \sum_{i\in A}(r-r_i)$, and so we have the conclusion: $Z = b - (X+Y) \geq \sum_{i\in A}(r - r_i) \geq 0$.
(2). Since $b + c = (r+1)(c-\delta)$, (2) follows easily. $\diamond$

**Lemma 2.3** Case $k = 2$. *With Setting 1.6, assume $k = 2$.*

(1) *If* $r = e-1$, *then* $b = (l+1)e + h \leq (l+2)e - 2$.

*Further:* $b = (l+2)e - 2 \iff h = e - 2$.

(2) *If* $r = e - 2$, *then:*

$b = (l+1)(e-1) + h - p - 1,$

$c = (p + l + 2)(e-1) - b.$

*Further we have:*

(a) $l + 1 \leq p \leq 2l + 2$ *and* $p = 2l + 2 \Longrightarrow h > 0$.

(b) $(l+1)(e-3) \leq b \leq (l+1)(e-2) + e - 3$. *In particular*

$b = (l+1)(e-3) \iff p = 2l+2, h = 1$ *or* $p = 2l+1, h = 0$.

$b = (l+1)(e-2) + e - 3 \Longrightarrow p = l+1,\ p > 1,\ h = e-2,\ y = e+1.$

Proof. For $k = 2$, we write $v(R) = \{0, e, 2e, ..., pe, c, \rightarrow\} \cup \{y, y+e, ..., y+le\}$, with $r \in \{e-2, e-1\}$, $c - \delta = p + 2 + l$, $c = (p+1)e - h$. (see 1.8), (1.6)). Then the expressions of $b$, in items 1,2, come from (1.8.4) with $k = 2$ and $e - r = 1, 2$, respectively. To complete the proof of item 1 recall that $h \leq e - 2$.
The bounds for $p$ in item 2 come from (1.8.2) and (1.11.3) and the value of $c$ comes from (2.2.2).
Rewriting $b$ in the form

$b = (l+1)(e-2) + (l-p) + h,$

and recalling that $l - p \leq -1,\ h \leq e-2$, we obtain the upper bound for $b$.
Rewriting $b$ in the form

$b = (l+1)(e-3) + (2l+2-p) + (h-1)$,
and using part $a$, we obtain the lower bound and also $b = (l+1)(e-3) \iff p = 2l+2, h = 1$ or $p = 2l+1, h = 0$.
Finally, note that $b = (l+1)(e-2) + e - 3 \Longrightarrow h = (p-l-1) + e - 2 \geq e - 2 \Longrightarrow p = l+1,\ h = e-2 \Longrightarrow c = pe+2$ and since by definition of $l$ $y + le < c$, it follows that $y < e+2$, hence $y = e+1$ and $p > 1$. $\diamond$

**Lemma 2.4** Case $k = 3$. *With Setting 1.6, assume $k = 3$.*

(1) *If $r = e-3$, then $b = (l_1+l_2+2)(e-2) + h - 2(p+1)$. Moreover,*
$p < l_1+l_2+2 \Longrightarrow b \geq (l_1+l_2+2)(e-4) + h,\ h \geq 0.$
$p = l_1+l_2+2 \Longrightarrow b = (l_1+l_2+2)(e-4) + h - 2,\ h > 0.$

(2) *If $r = e-2$, then $b = (l_1+l_2+2)(e-1) + h - p - 1$ and $p \leq 2l_1 + 2$. Further, $p = 2l_1+2 \Longrightarrow h > 0$.*

(3) *If $r = e-1$, then $b = (l_1+l_2+2)e + h$.*

Proof. Recall that by (1.4) $e - r \leq 3$ and by (1.6)
$v(R) = \{0, e, 2e, ..., pe, c, \rightarrow\} \cup \{y_1, ..., y_1 + l_1 e\} \cup \{y_2, ..., y_2 + l_2 e\}$.
Formula (2.2.1) with $k = 3$ becomes
$b = (r+1)(l_1+l_2+2) - (p+1)(e-r-1) + h$.
By substituing $r$ with $e-1, e-2, e-3$, we get the desired expressions for $b$ in items 3, 2, 1, respectively. To complete the proof of (1) and (2), apply (1.11.3) and (1.11.2), respectively. $\diamond$

**Proposition 2.5** *Let $q \in \mathbb{N}$ be such that $0 < b \leq q(r-1)$. Then:*

(1) *$r \geq e - q - 1$. In particular,*

- (a) *If $r = e-1-q$, then $b = q(r-1), q \leq e-3$, and the equivalent conditions of (2.1.4) hold.*
- (b) *If $r \geq e-q$, then $e - r \leq k \leq q$.*

(2) *If $(q-1)(r-1) < b \leq q(r-1)$, we have:*

- (c) *$k - 1 \leq q \leq n - 1$.*
- (d) *$(q-k-1)(r-1) < \sum_{i \in A}(r - r_i) \leq (q-k)(r-1) + e - 1 - k$.*

Proof. (1). First we deduce the inequalities
$$(\odot) \qquad (e-r-1)(r-1) \leq b \leq q(r-1),$$
by combining (2.1.1) with the assumption. Hence we get $r \geq e - q - 1$.
($a$). If $r = e-1-q$, then relations ($\odot$) give $b = (e-r-1)(r-1)$, and so the conditions of (2.1.4) hold.
($b$). Assertion ($**$) in the proof of (2.1) insures that $kr - (e-1) \leq b$. Hence assuming $r \geq e-q$, we have $rk \leq b + e - 1 \leq q(r-1) + e - 1 \leq (q+1)r - 1$; then $k \leq q$.
(2). Put $M := \sum_{i \in A}(r - r_i)$. We have to compare the two inequalities of (2.1.1)

$(k-1)(r-1)+M+k-(e-r) \leq b \leq k(r-1)+M$
with the assumption
$(q-1)(r-1) < b \leq q(r-1)$.
We obtain the following:
$(k-1)(r-1)+M+k-(e-r) \leq q(r-1)$, and also
$(q-1)(r-1) < k(r-1)+M$.
The first inequality gives $q \geq k-1$ and $M \leq (q-k)(r-1)+e-1-k$.
The second one says that $M > (q-k-1)(r-1)$. Moreover, combining the hypothesis with (2.1.2)
$(q-1)(r-1) < b \leq (n-1)(r-1)$,
and this implies $q \leq n-1$, as desired. $\diamond$

# 3 Classification.

Our aim is now to classify the value sets for one dimensional local domains having

$$0 \leq b \leq 2(r-1).$$

On this topic several results are present in the literature. For semigroup rings $R = k[[t^\alpha; \alpha \in S]]$, $S \subset \mathbb{N}$ a numerical semigroup, Brown and Herzog in [2, Corollary after Theorem 4] illustrate the case $b = 0$. This result can be extended to rings $R$ as in Setting 1.1 (see (3.1)). Successively D. Delfino in [5, Corollary 2.11 and Corollary 2.14] gives a characterization of rings satisfying the condition $b < r-1$ and exhibits all the possible value sets in the case $b \leq r$, under the additional assumption $r = e-1$. See also Proposition 2.7 from [3] for a further generalization. An exaustive description of the cases $0 < b < r-1$ can be found in [10, Th.4.6].

In this section we assume Setting 1.1 and Notation 1.3. Moreover, $t.s.(R)$ will denote the type sequence of $R$, defined in (1.1).

First we recall in (3.1) and (3.2) the quoted known results, which now become an easy consequence of our preceding statements.

**Theorem 3.1** Case $b = 0$.
*The following conditions are equivalent:*

(1) $b = 0$.

(2) *Either $R$ is Gorenstein, or* $v(R) = \{0, e, .., pe, (p+1)e \rightarrow\}$.

(3) $t.s.(R) = [r, ..., r]$.

Proof. By (2.1.1) $0 = b \geq (k-1)(r-1)$; hence either $r = 1$ or $k = 1$, and this last condition gives, by (1.5), $v(R) = \{0, e, .., pe, (p+1)e \rightarrow\}$, or equivalently, $t.s.(R) = [e-1, ..., e-1] = [r, .., r]$. Hence $1 \Longrightarrow 2 \Longleftrightarrow 3$ are clear. Of course, in the Gorenstein case we have $t.s.(R) = [1, .., 1]$. Implication $3 \Longrightarrow 1$ is immediate by (1.2.1).

**Theorem 3.2** [10, Theorem 4.6.1] Case $0 < b < r-1$.
*The following facts are equivalent:*

(1) $0 < b < r-1$.

(2) $v(R) = \{0, e, .., pe, c \rightarrow\}$ *with* $pe + 2 < c \leq (p+1)e$.

(3) $ts(R) = [e-1, e-1, ..., e-1, r_n]$, $r_n > 1$.

*If these conditions hold, then:*
$b < e-2$, $r = e-1$, $r_n = e-1-b$, $k = 1$, $c = (p+1)e - b$.

**Theorem 3.3** Case $b = r-1$.
*If* $b = r-1 > 0$, *then either* $r = e-1$ *or* $r = e-2$.

*1.* Subcase $r = e-1$. *The following facts are equivalent:*

*(a)* $b = r-1 > 0$ *and* $r = e-1$.
*(b)* $v(R) = \{0, e, .., pe, pe+2 \rightarrow\}$, $e > 2$.
*(c)* $ts(R) = [e-1, ...., e-1, 1]$, $e > 2$.
*(d)* $b = r-1 > 0$ *and* $k = 1$.

*2.* Subcase $r = e-2$. *The following facts are equivalent:*

*(e)* $b = r-1 > 0$ *and* $r = e-2$.
*(f)* *either* $v(R) = \{0, e, 2e-1, 2e, 3e-1 \rightarrow\}$, $e > 3$,
*or* $v(R) = \{0, e, y, 2e \rightarrow\}$, *with* $2y < 3e$, $e > 3$.
*(g)* *either* $ts(R) = [e-2, e-2, 1, e-2]$, *with* $e > 3$,
*or* $ts(R) = [e-2,\ r_2,\ r_3]$, *with* $r_2 + r_3 = e-1$, $e > 3$.
*(h)* $b = r-1 > 0$ *and* $k = 2$.

Proof. Applying (2.5.1) with $q = 1$, we obtain that $r \geq e-2$. Further, if $b = r-1$, then $r = e-2 \Longleftrightarrow k = 2$ by (2.5.1$a$) and (2.1.4); also, if $b = r-1$, then $r = e-1 \Longleftrightarrow k = 1$ by (2.5.1$b$). This proves the first assertion and the equivalences $a \Longleftrightarrow d$, $e \Longleftrightarrow h$.

(1). First note that $(a)$ implies $e > 2$; in fact, $e = 2$ would imply $r = 1, b = 0$.
$d \Longrightarrow b$. Since $k = 1$, the equivalent conditions of (1.5) hold, and
$v(R) = \{0, e, .., pe, c \rightarrow\}$, with $c = (p+1)e - b = pe + 2, e > 2$.
$b \Longrightarrow c$. If $(b)$ holds, then by (1.5) $ts(R) = [e-1, ...., e-1, r_n]$, with $r - r_n = b = r-1$, hence $r_n = 1$, as in $(c)$.
$c \Longrightarrow a$. By (1.2.1), $(c)$ implies $r = e-1$ and $b = r - r_n = r-1$, as in $(a)$.

(2). First note that $(a)$ implies $e > 3$; in fact, $e = 3$ would imply $r = 1, b = 0$.
$h \Longrightarrow f$. Since $k = 2$ we use (2.3.2) recalling that $p \leq 2l + 2$:
$e - 3 = b = (l+1)(e-1) + h - p - 1 \geq (l+1)(e-1) + h - 2l - 3$.
Hence we get $l(e-3) + h \leq 1$ and the following possibilities occur by (2.3.2$c$):

$(l,p,h)=(0,1,0)$, or $(l,p,h)=(0,2,1)$, or $h=0,\ e=4,\ l=1$.

- If $(l,p,h)=(0,1,0)$, then $c=2e,\ v(R)=\{0,e,y,2e\rightarrow\}$ with $2y<3e,\ e>3$.
- If $(l,p,h)=(0,2,1)$, then $v(R)=\{0,e,2e,c\rightarrow\}\cup\{y\}$, with $c-\delta=4$, $c=(p+1)e-h=3e-1,\ 2y<c+e=4e-1\Longrightarrow y\le 2e-1$, $c-e\in v(R)\Longrightarrow y=2e-1$. Hence $v(R)=\{0,e,2e-1,2e,3e-1\rightarrow\},\ e>3$.
- If $h=0,\ e=4,\ l=1$, then $e-3=b=(l+1)(e-1)+h-p-1 \Longrightarrow$ $p=4=2l+2\Longrightarrow h>0$, which is absurd. Hence $h\Longrightarrow f$ is proved.

$f\Longrightarrow g$. Denoting $R_0=k[[t^d,\ d\in v(R)]]$ the monomial ring such that $v(R_0)=v(R)=\{0,e,2e-1,2e,3e-1\rightarrow\}$, we have $r(R_0)=e-2$. Since $r(R)\le r(R_0)$ and $r(R)\ge e-2$ by (1.4.1), we conclude that $r(R)=e-2$. The other invariants are easily derived from $v(R)$: $c-\delta=4,\ \delta=3e-5,\ b=(c-\delta)r-\delta=e-3$. By substituting in (2.1.1), we obtain $\sum_{h\in A}(r-r_h)=0$, hence $r_2=e-2$ and $r_3+r_4=e-1$, as desired.
The same reasoning holds for $v(R)=\{0,e,y,2e\rightarrow\}$.
To see $g\Longrightarrow e$, it suffices to recall that $b=\sum_{h=1}^{n}(r-r_h)$, see (1.2.1). $\diamond$

**Theorem 3.4** Case $r-1<b<2(r-1)$.
*We have* $r-1<b<2(r-1)$ *if and only if* $v(R)$ *is one of the following:*

(1) $v(R)=\{0,e,....,pe,c\rightarrow\}\cup\{y\}$, *with* $y\notin e\mathbb{Z}$,
*and either* $2y\ge c+e,\ pe+5\le c\le min\{y+e,(p+1)e\},\ e\ge 5$,
*or* $e=2e',\ y=3e',\ p=2,\ 4e'+5\le c\le 5e',\ e\ge 10,\ y\in v(xR:\mathfrak{m})$.

(2) $v(R)=\{0,e,2e,c\rightarrow\}\cup\{y\}$, *with* $y\notin e\mathbb{Z},\ 2y<c+e$ *and:*
*if* $2y\ne 3e$, *then* $2e+3\le c\le 3e-2,\ e\ge 5$;
*if* $2y=3e$, *then* $e=2e',\ 4e'+3\le c\le 5e',\ e\ge 6,\ y\notin v(xR:\mathfrak{m})$.

(3) $v(R)=\{0,e,y,c\rightarrow\}$,
*with* $y\notin e\mathbb{Z},\ e\ge 5,\ 2y<c+e,\ e+4\le c\le 2e-1$.

*In each case* $k=2$; *in case* (1), $r=e-1$ *and* $b\ge r+1$; *in cases* (2) *and* (3), $r=e-2$.

Proof. Assume $r-1<b<2(r-1)$.
Step 1. Claim: $k=2$ and $e-2\le r\le e-1,\ r>2$.
We have $r>2$, since $r=2\Longrightarrow 1<b<2$, which is absurd. Further (2.1.1) gives $(k-1)(r-1)\le b$, and so $k\le 2$. But $k=1$ would imply $b\le r-1$ by (1.5), then $k=2$. We conclude using (1.4.1).
Now utilizing the notation in (1.6) we write:

$$(*)\begin{cases} v(R)=\{0,e,2e,...,pe,c,\rightarrow\}\cup\{y,y+e,...,y+le\},\ p\ge 1,\ y>e,\ y\notin e\mathbb{Z},\\ y+le<c=(p+1)e-h\le y+(l+1)e,\ \ l+1\le p.\end{cases}$$

Step 2. Claim: $l=0$ and $e\ge 5$. Further, if $r=e-2$, then $p\le 2$.

- If $r=e-1$, then, by (2.3.1) we know that $b=(l+1)e+h,\ l,h\ge 0$. Hence $b<2(r-1)=2e-4\Longrightarrow (l-1)e+h<-4\Longrightarrow\ l=0,\ h<e-4$; further we get : $c=(p+1)e-h\ge pe+5,\ e\ge 5$ and $b=h+e\ge e=r+1$.
- If $r=e-2$, we have $(l+1)(e-3)\le b$ and $l+1\le p\le 2l+2$ by (2.3.2). Then $b<2(r-1)=2(e-3)\Longrightarrow l=0$ and $p\le 2$; also, the assumption $e-3<b<2e-6$ implies $e\ge 5$.

Step 3. When $r = e - 1$, recalling the relations proved in Step 2, we obtain $v(R) = \{0, e, ...., pe, c \rightarrow\} \cup \{y\}$, with $e \geq 5$, $pe + 5 \leq c$, as in item 1. Recall that by definition of $p$ and $l$ we have $c \leq (p+1)e$ and $c \leq y + e$. Moreover, by (1.12.1) one of the following conditions is satisfied:
either $(a)$ $2y \geq c + e$, or $(b)$ $2y = (2q+1)e < c + e$, $p \geq 2$ and $y \in v(xR : \mathfrak{m})$.
Further, as noted in (1.11.4), $p \geq 3$, $l = 0 \Longrightarrow 2y > c + e$, hence in case $(b)$ we have $p = 2$ and consequently $(2q+1)e < c + e \leq 4e \Longrightarrow q = 1$. This proves (1).

Step 4. When $r = e - 2$, we have by Step 2 that $l = 0$ and $p \leq 2$.
In the case $p = 2$ we get item 2. In fact from (2.3.2) we obtain $c = 4e - 4 - b$ and the bounds for $c$ follow at once. The last assertion in item 2 comes from (1.12). Analogously, in the case $p = 1$ we get item 3. Notice that when $p = 1$ we cannot have $2y = 3e < c + e$, because $c + e \leq 3e - 1$.

To complete the proof, let $v(R)$ be as in items $(1), (2), (3)$; we claim that $r - 1 < b < 2(r-1)$. In every case $k = 2$; in case (1) $r = e - 1$ and in cases (2), (3) $r = e - 2$ by (1.12). The rest is a direct computation based on relation (2.2.2): $c = (p+2)(r+1) - b$. $\diamond$

**Example 3.5** We supply an example for each case of the above proposition.

- Case (1) with $2y \geq c + e$.

Let $R = k[[t^5, t^{10}, t^{12}, t^{15}, \rightarrow]]$. Then: $y = 12, p = 2, c = 15, r = 4, b = 5$.

- Case (1) with $2y = 3e$.

Let $R = k[[t^{10}, t^{15}, t^{20}, t^{25}, \rightarrow]]$. Then: $y = 15, p = 2, c = 25, r = 9, b = 15$.

- Case (2).

Let $R = k[[t^{10}, t^{15} + t^{16}, t^{20}, t^{25}, \rightarrow]]$. As above, $y = 15, p = 2, c = 25$, but $r = 8$ by 1.9 since $(t^{15} + t^{16})^2 \notin x\mathfrak{m}$. Then $b = 11$.

- Case (3).

Let $R = k[[t^5, t^6, t^9, \rightarrow]]$. Here $y = 6, p = 1, c = 9, r = 3, b = 3$.

**Theorem 3.6** Case $b = 2(r-1)$.
$b = 2(r-1) > 0$ *if and only if* $v(R)$ *is one of the following:*

1. *(a)* $v(R) = \{0, e, e+2, e+4 \rightarrow\}$, $e \geq 4$.
   *(b)* $v(R) = \{0, e, 2e, 2e+4 \rightarrow\} \cup \{y\}$, $e \geq 4$, $y \in v(xR : \mathfrak{m})$.
   *(c)* $v(R) = \{0, e, 2e, ..., pe, pe+4 \rightarrow\} \cup \{y\}$, $e \geq 4$, $y \geq (p-1)e+4$, $p \geq 3$.
2. *(a)* $v(R) = \{0, e, e+1, e+3 \rightarrow\}$, $e \geq 4$.
   *(b)* $v(R) = \{0, e, y, 2e, 2e+2 \rightarrow\}$, $e \geq 5$, $2e+4 \leq 2y < 3e+2$, $2y \neq 3e$.
   *(c)* $v(R) = \{0, e, 2e, 3e-1, 3e, 4e-1, 4e, 5e-1 \rightarrow\}$, $e \geq 4$.
   *(d)* $v(R) = \{0, e, 2e, y, 3e, y+e, 4e \rightarrow\}$, $e \geq 4$, $2y < 5e$.
3. *(a)* $v(R) = \{0, e, y_1, y_2, 2e \rightarrow\}$, $e \geq 5$, $y_1 + y_2 < 3e$.
   *(b)* $v(R) = \{0, e, 2e-2, 2e-1, 2e, 3e-2 \rightarrow\}$, $e \geq 5$.

*In case 1* $r = e - 1$ *and* $\ell_R(R/(\mathfrak{C} + xR)) = 2$, *in case 2* $r = e - 2$ *and* $\ell_R(R/(\mathfrak{C} + xR)) = 2$, *in case 3* $r = e - 3$ *and* $\ell_R(R/(\mathfrak{C} + xR)) = 3$.

Proof. Let, as above, $k = \ell_R(R/(\mathfrak{C}+xR))$. First we assume $b = 2(r-1) > 0$ and we observe that by (2.1.1) $(k-1)(r-1) \leq b = 2(r-1)$, then $k \leq 3$. Since $k = 1$ implies $b \leq r-1$ by (1.5), one of the following cases occurs:

$$\left[\begin{array}{llll} or & k=2 & and & r=e-1 \\ or & k=2 & and & r=e-2 \\ or & k=3 & and & r=e-3. \end{array}\right.$$

In case $k = 2$ by Setting 1.6 we have:

$$(*)\begin{cases} v(R)=\{0,e,2e,...,pe,c,\rightarrow\}\cup\{y,y+e,...,y+le\},\ p\geq 1,\ y>e,\ y\notin e\mathbb{Z}, \\ y+le<c=(p+1)e-h\leq y+(l+1)e,\quad l+1\leq p. \end{cases}$$

Step 1. Assuming $r = e-1$ and $k = 2$, we prove that $v(R)$ has the form described in item 1. By (2.3.1) and the assumption we have the equalities $b = (l+1)e+h = 2e-4$; hence $(l-1)e+h = -4 \Longrightarrow l = 0,\ h = e-4,\ e \geq 4,\ c = (p+1)e-h = pe+4$. Now, $l = 0 \Longrightarrow y \geq c-e = (p-1)e+4$, and so

$$v(R)=\{0,e,...,pe,pe+4\rightarrow\}\cup\{y\},\ \text{ with }\ (p-1)e+4\leq y\leq pe+2,\quad e\geq 4.$$

For $p = 1$ we get (1.$a$). In fact, by (1.12.1) $2y \geq c+e = 2e+4 \Longrightarrow y \geq e+2 \Longrightarrow y = e+2$. For $p = 2$ we get (1.$b$). For $p \geq 3$ we get (1.$c$).

Step 2. Assuming $r = e-2$ and $k = 2$, we prove that $v(R)$ satisfies item 2. First, by (2.3.2) we have that $l+1 \leq p \leq 2l+2$ and also that

$$(**)\quad (l+1)(e-3)\leq (l+1)(e-1)+h-p-1=b.$$

Then $b = 2(e-3) > 0$ implies $(l+1)(e-3) \leq 2(e-3)$, i.e. $l \leq 1$.

Case $l = 0$, and consequently $1 \leq p \leq 2$.

($\cdot$) If $l = 0,\ p = 1$, then by $(**)$, $h = e-3$, thus $c = e+3$, and (2.$a$) holds.

($\cdot$) If $l = 0,\ p = 2$, then $h = e-2, c = 2e+2$, hence (2.$b$) holds.

Case $l = 1$. Now, relation $(**)$ combined with the assumption $b = 2e-6$ implies $h-p-1 = -4,\ 2 \leq p \leq 4$ and two possibilities occur:

($\cdot$) $p = 4,\ h = 1,\ c = 5e-1$. The relation $c \leq y+(l+1)e$ gives $y \geq 3e-1$, the relation $2y < c+e$ gives $y \leq 3e-1$. Hence (2.$c$) holds.

($\cdot$) $p = 3,\ h = 0,\ c = 4e$; hence (2.$d$) holds.

Step 3. Assuming $r = e-3$ and $k = 3$, we prove that $v(R)$ has the form described in item 3. First, by Setting 1.6 and by (2.4.1) we have:

$$(\overline{*})\begin{cases} v(R)=\{0,e,...,pe,c\rightarrow\}\cup\{y_1,y_1+e,...,y_1+l_1e\}\cup\{y_2,y_2+e,...,y_2+l_2e\} \\ p\geq 1,\ y_2>y_1>e,\ y_i\notin e\mathbb{Z}, \\ y_i+l_ie<c=(p+1)e-h\leq y_i+(l_i+1)e,\quad l_i+1\leq p, \\ b=(l_1+l_2+2)(e-2)+h-2(p+1). \end{cases}$$

By (1.11.3), since $r = e-k$, then $p \leq l_1+l_2+2$.

($\cdot$) If $p < l_1+l_2+2$, then substituting $b = 2(e-4) > 0$ in $(\overline{*})$ we get $(l_1+l_2)(e-4)+h \leq 0,\ h \geq 0$. Hence $h = l_1 = l_2 = 0,\ p = 1,\ c = 2e,\ y_1+y_2 < c+e$ by (1.11.3), and so we have (3.$a$).

($\cdot$) If $p = l_1+l_2+2$, then analogously we get $(l_1+l_2)(e-4)+h-2 = 0$, with $0 < h \leq 2$. The case $h = 1$ is impossible. In fact, $h = 1 \Longrightarrow l_1+l_2 = 1$ (in particular, by (1.8.2), $l_2 \leq l_1$, hence $l_2 = 0,\ l_1 = 1$), $e = 5,\ p = 3,\ c = (p+1)e-h = 19$. The relation of (1.6) $c \leq y_i+(l_i+1)e$ gives $y_1 \geq 19-10 = 9,\ y_2 \geq 19-5 = 14$, but $y_1+y_2 < c+e = 24$ by (1.11.3); the only possibility would be $y_1 = 9,\ y_2 = 14$. Absurd that $\overline{y_1} = \overline{y_2}\ (mod\ 5)$. Hence $h = 2,\ l_1 = l_2 = 0,\ p = 2,\ c = 3e-2$ and

$$v(R)=\{0,e,2e,3e-2,\rightarrow\}\cup\{y_1,y_2\}.$$

Since $l_1 = 0$, the bound $c \leq y_1+e$ gives $y_1 \geq 2e-2$. Recalling that by (1.11.3)

$y_1 + y_2 < c + e$, we conclude $y_1 = 2e - 2, y_2 = 2e - 1$, as in $(3.b)$.

Viceversa, we assume in the following $v(R)$ having the form described in items 1,2,3, and we prove that $b = 2(r-1) > 0$.
For a $v(R)$ as in item 1 we see that $r = e - 1$ using (1.12). In fact, in case $(1.a)$ we have $y = e + 2,\ 2y = c + e$ and in case $(1.c)$:

$2y \geq 2(p-1)e + 8 > c + e = (p+1)e + 4$.

In conclusion in each case of item 1 we have $\ell_R(R/(\mathfrak{C}+xR)) = 2,\ r = e-1,\ l = 0$. Using (2.3.1) $b = e + h = 2e - 4 = 2(r-1)$, as desired.
In case $(2.a)$, $y = e + 1 \notin v(xR : \mathfrak{m})$, then $r = e - 2$ by (1.9). In case $(2.b)$ by hypothesis $2y < c + e$ and $2y \neq 3e$, then $r = e - 2$ by (1.12). In case $(2.c)$ we get by a direct calculation $v(xR_0 : \mathfrak{m}) \setminus v(R_0) = \{4e+1, ...., 5e-2\}$, then $r = r(R_0) = e-2$. In case $(2.d)$ $2y < c+e$ and $2y \notin e\mathbb{Z}$, then $r = e-2$ by (1.12). In conclusion, in each case of item 2 one has: $\ell_R(R/(\mathfrak{C} + xR)) = 2,\ r = e - 2$, and so by (2.3.2) $b = (l+1)e - 1) + h - p - 1$. Putting in this formula

$(\cdot)$ $l = 0,\ p = 1,\ h = e - 3$, in case $(2.a)$,

$(\cdot)$ $l = 0,\ p = 2,\ h = e - 2$, in case $(2.b)$,

$(\cdot)$ $l = 1,\ p = 4,\ h = 1$, in case $(2.c)$,

$(\cdot)$ $l = 1, p = 3,\ h = 0$, in case $(2.d)$,

we get $b = 2e - 6 = 2(r-1)$, as desired.
In both cases of item 3 we have $r = e-3$. In fact, $y_1+y_2-e \notin v(R) \Longrightarrow y_1, y_2 \notin v(xR : \mathfrak{m}) \Longrightarrow e-r = 3$ by (1.11.1). Hence $\ell_R(R/(\mathfrak{C}+xR)) = 3,\ r = e-3,\ l_1 = l_2 = 0$, and by (2.4.3) $b = 2(e-2) + h - 2(p+1)$. Putting in this formula

$(\cdot)$ $h = 0,\ p = 1$ in case $(3.a)$,

$(\cdot)$ $h = p = 2$ in case $(3.b)$,

we get $b = 2e - 8 = 2(r-1)$, as desired. $\diamond$

With similar arguments one can evaluate the semigroups $v(R)$ of rings having $b > 2(r-1)$. For instance, if $2(r-1) < b \leq 3(r-1)$ there are few possible cases and the classification is tedious but easy. Now, for each $q \geq 3$ we construct a family of rings of multiplicity $e$ and Cohen Macaulay type $r = e - 1$ having $b = q(r-1)$ or $(q-1)(r-1) < b < q(r-1)$.

**Example 3.7** Let $q \geq 3$. Following notations of Setting 1.6 we consider

$v(R) = \{0, e, 2e, ..., pe, c \rightarrow\} \cup \{y, y+e, ..., y+le\}$,

with $e > p,\ p = 2q,\ l = q - 2$. In this case $k = 2$. Using (1.12) we see that $r = e-1$, because $y + (q-1)e \geq c > 2qe \Longrightarrow y > (q+1)e \Longrightarrow 2y > 2(q+1)e \geq c + e$. Then by (2.3.1) $b = (q-1)e + h$, with $0 \leq h \leq e - 2$. Now, with an additional hypothesis on the conductor, we are in goal. In fact:

1) Assuming $c = pe + p$, we have $h = (p+1)e - c = -p + e = -2q + e$, then $b = (q-1)e + (-2q+e) = q(e-2) = q(r-1)$.

2) Assuming $c > pe+p$, i.e. $e-h > 2q$, we have $(q-1)(e-2) < (q-1)e \leq b = (q-1)e + h = q(e-2) + 2q - e + h < q(e-2)$, hence $(q-1)(r-1) < b < q(r-1)$.

As a further application of the previous results we describe exhaustively the cases $b = 1$ and $b = 2$ (see next (3.8), (3.9); for $b = 1$ see also [2], Section 4).

With regard to the formula

$$b = \sum_{i=1}^{n} (r - r_i)$$

it becomes natural to consider the invariant $b$as a measure of how far is the type sequence $[r_1, ..., r_n]$ from the maximal one $[r, ..., r]$. For instance, for $b = 1$ one expects a type sequence of the form $[r, ..., r-1, ..., r]$, for $b = 2$ $[r, ..., r-1, ..., r-1, ..., r]$ or $[r, ..., r-2, ..., r]$, and so on. Surprisingly, after finding by a direct computation all the possible value sets and the corresponding type sequences, we discover that very few choices are possible. For $b = 1$ (resp. $b = 2$) either $e \leq 4$ (resp. $e \leq 5$) or $t.s.(R) = [e-1, ..., e-1, e-1-b]$.

In the following t.s. stands for type sequence.

**Corollary 3.8** Case $b = 1$.
$b = 1$ *if and only if* $v(R)$ *is one of the following:*
$v(R) = \{0, 4, 7, 8, 11 \rightarrow\}$, *with t.s.* $[2, 2, 1, 2]$;
$v(R) = \{0, 4, 5, 8, \rightarrow\}$, *with t.s.* $[2, 1, 2]$;
$v(R) = \{0, e, ..., pe, (p+1)e - 1, \rightarrow\}$, $e \geq 3$, *with t.s.* $[e-1, ..., e-1, e-2]$.

Proof. First recall that $b > 0 \Longrightarrow r > 1$ by (1.2.1). Let, as in (2.2.1), $b = X + Y + Z$, where $X := (k-1)(r-1) \geq 0$, $Y := k - (e-r) \geq 0$, and $Z := (r+1)(p + \sum_1^{k-1} l_i) + k + h - pe - 1 \geq 0$.
Assuming $b = 1$, we have to consider the choices:

| | $X$ | $Y$ | $Z$ |
|---|---|---|---|
| $a)$ | 1 | 0 | 0 |
| $b)$ | 0 | 1 | 0 |
| $c)$ | 0 | 0 | 1 |

In $a)$ $k = r = 2$, $2 - (e-2) = 0 \Longrightarrow e = 4$. By (3.3.2) with $e = 4$ we find:
$v(R) = \{0, 4, 7, 8, 11 \rightarrow\}$,
$v(R) = \{0, 4, 5, 8, \rightarrow\}$.
In $b)$ $k = 1$, $1 - (e-r) = 1$, which is absurd.
In $c)$ $k = 1$, $1 - (e-r) = 0 \Longrightarrow r = e-1$, $e \geq 3$, $Z = ep + 1 + h - pe - 1 = 1 \Longrightarrow h = 1 \Longrightarrow c = (p+1)e - 1$. By (1.5) we find:
$v(R) = \{0, e, ..., pe, (p+1)e - 1, \rightarrow\}$, $e \geq 3$. $\diamond$

**Corollary 3.9** Case $b = 2$.
$b = 2$ *if and only if* $v(R)$ *is one of the following:*
$v(R) = \{0, 4, 5, 7, \rightarrow\}$, *with t.s.* $[2, 1, 1]$;
$v(R) = \{0, 4, 8, 11, 12, 15, 16, 19, \rightarrow\}$, *with t.s.* $[2, 2, 2, 1, 2, 1, 2]$;
$v(R) = \{0, 4, 8, 9, 12, 13, 16, \rightarrow\}$, *with t.s.* $[2, 2, 1, 2, 1, 2]$;
$v(R) = \{0, 5, 9, 10, 14, \rightarrow\}$, *with t.s.* $[3, 3, 1, 3]$;
$v(R) = \{0, 5, 6, 10, \rightarrow\}$, *with t.s.* $[3, 1, 3]$;
$v(R) = \{0, 5, 7, 10, \rightarrow\}$, *with t.s.* $[3, 2, 2]$;
$v(R) = \{0, 5, 6, 7, 10, \rightarrow\}$, *with t.s.* $[2, 1, 1, 2]$;
$v(R) = \{0, 5, 6, 8, 10, \rightarrow\}$, *with t.s.* $[2, 2, 1, 1]$;
$v(R) = \{0, 5, 8, 9, 10, 13, \rightarrow\}$, *with t.s.* $[2, 2, 1, 1, 2]$;

$v(R) = \{0, e, ..., pe, (p+1)e-2, \rightarrow\},\ e \geq 4,$ *with t.s.* $[e-1, ..., e-1, e-3]$.

Proof. As in the preceding proof, assuming $b = 2$, we have to consider the following choices:

| | $X$ | $Y$ | $Z$ |
|---|---|---|---|
| $a)$ | 0 | 1 | 1 |
| $b)$ | 1 | 0 | 1 |
| $c)$ | 1 | 1 | 0 |
| $d)$ | 2 | 0 | 0 |
| $e)$ | 0 | 2 | 0 |
| $f)$ | 0 | 0 | 2 |

First recall that $k = 1 \Longrightarrow r = e - 1$ by (1.5), and so $X = 0$ (with $r > 0$) $\Longrightarrow k - (e - r) = Y = 0$ and cases $a), e)$ are impossible.
In $b)$ $X = 1 \Longrightarrow k = r = 2,\ 2 - (e-2) = Y = 0 \Longrightarrow e = 4$, hence $b = 2(r-1)$ and we can apply (3.6.2) with $e = 4$. We find:
$v(R) = \{0, 4, 5, 7, \rightarrow\}$,
$v(R) = \{0, 4, 8, 11, 12, 15, 16, 19, \rightarrow\}$,
$v(R) = \{0, 4, 8, 9, 12, 13, 16, \rightarrow\}$.
In $c)$ $X = 1 \Longrightarrow k = r = 2,\ 2-(e-2) = Y = 1 \Longrightarrow e = 3,\ Z = 3(p+l)+2+h-3p-1 = 0 \Longrightarrow 3l + h + 1 = 0$, which is absurd.
In $d)$ the condition $X = (k-1)(r-1) = 2$ implies two possibilities:
$d_1)$ $k = 2,\ r = 3,\ 2-(e-3) = 0 \Longrightarrow e = 5$. We are in case $b = r-1,\ r = e-2$. By (3.3.2) with $e = 5$ we find:
$v(R) = \{0, 5, 9, 10, 14, \rightarrow\}$,
$v(R) = \{0, 5, 6, 10, \rightarrow\}$,
$v(R) = \{0, 5, 7, 10, \rightarrow\}$.
$d_2)$ $k = 3,\ r = 2,\ e = 5$. We are in case $b = 2(r-1),\ r = e-3$, and so by (3.6.3) with $e = 5$ we find:
$v(R) = \{0, 5, 6, 7, 10, \rightarrow\}$,
$v(R) = \{0, 5, 6, 8, 10, \rightarrow\}$,
$v(R) = \{0, 5, 8, 9, 10, 13, \rightarrow\}$.
In $f)$ $k = 1,\ r = e-1,\ Z = ep+1+h-pe-1 = 2 \Longrightarrow h = 2 \Longrightarrow c = (p+1)e-2$. By (1.5) we find:
$v(R) = \{0, e, ..., pe, (p+1)e-2, \rightarrow\},\ e \geq 4.\quad \diamond$

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