ON THE VALUE SET OF MODULES.

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Let $(R, \mathfrak{m}_R)$ be the local ring of a reduced and irreducible analytic curve singularity, with value semigroup Γ, conductor c and normalization $\overline{R}$. To a given isomorphism class of torsion free R-modules of rank 1 we can relate a value set $\Gamma(M)$ by choosing a representative M such that $R \subset M \subset \overline{R}$. It is a well known fact that the class of the canonical ideal $\widetilde{\omega}_R$ is determined by its value set $\{j \in \mathbb{N} \mid c-1-j \notin \Gamma\}$. Starting from this remark we consider for any integer c_k such that $c_k-1 \notin \Gamma$ the numerical set $L_k := \{j \in \mathbb{N} \mid c_k-1-j \notin \Gamma\}$. For any R-module as above the immersion $R \subset M \subset \overline{R}$ assures that $\Gamma(M)$ is contained in L_k, where $c_k = c(M)$ is the conductor of M. In particular the whole set L_k corresponds to a unique isomorphism class of R-modules, namely the one of the canonical ideal of a special overring of R. These simple observations allow us to state relations among various invariants associated with M or $\Gamma(M)$ and strictly connected with the Cohen-Macaulay type. Lower and upper bounds are discussed for the invariant $\delta(M:M)$, which represents the dimension of the orbit of M in the module variety $\mathcal{M}(R)$ [4]. Our result on the maximality of $\delta(M:M)$ can be written in terms of rings of maximal lenght and can be interpreted as a generalization for modules of an analogous statement for rings of W.C. Brown and J.Herzog [1]. In the last section some computations are given in the case $\Gamma(R)$ is a monomial semigroup, according to [9].

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Let $R = k[\![x_1, ..., x_n]\!]$ be a complete local k-algebra of dimension one with maximal ideal $\mathfrak{m}_R$, where k is an algebraically closed field of characteristic zero. We suppose R an integral domain. We denote $\overline{R} := k[\![t]\!]$ the integral closure of R in its quotient field $K := k\{\{t\}\}$ and $\nu : k\{\{t\}\} \longrightarrow \mathbb{Z} \cup \infty$ the canonical valuation given by the degree in t. Then:

$\Gamma := \Gamma(R)$ is the *value semigroup* of R;

$c := c(R)$ is the valuation of any generator of the conductor $\mathcal{C}$ as $\overline{R}$ -ideal;

$e := e(R)$ is the multiplicity of R;

$\delta := \delta(R) = \dim_k(\overline{R}/R)$ is the *singularity degree* of R.

We shall denote by $c_0 = 0 < c_1 = 2 < ... < c_\delta = c$ the integer numbers such that $c_k - 1 \notin \Gamma$ for all $k = 0, ..., \delta$. They correspond naturally to a chain of overrings: $\overline{R} = R_0 \supset R_1 \supset ...R_k... \supset R_\delta = R$ defined by $R_k := R + t^{c_k}\overline{R}$.

Given a fractional R-ideal $M \subset K$:

$\Gamma(M) := \{\nu(m) \mid m \in M\}$ = the image of M under ν is the *value set* of M; $\Gamma(M)$ is a Γ-set, i.e. $\Gamma(M) + \Gamma \subset \Gamma(M)$

$c(M) :=$ the smallest integer such that $t^{c(M)}\overline{R} \subset M$ is the *conductor* of M, i.e. $c(M) + \mathbb{N} \subset \Gamma(M)$ and $c(M) - 1 \notin \Gamma(M)$.

$\delta(M) := \dim_k(\overline{M}/M)$ where $\overline{M} := M \otimes_R \overline{R}/torsion$ is called the *δ-invariant* of M

Recall that for any submodule N of M the length of the R-module M/N can be computed by means of valuations: $l_R(M/N) = \#\ \Gamma(M) \setminus \Gamma(N)$ (see [6]) Since any finitely generated torsion free R-module of rank 1 is isomorphic to some fractional ideal such that $R \subset M \subset \overline{R}$ (see [4], Lemma 1.1), without loss of generality we shall assume in the following that

$$R \subset M \subset \overline{R}$$

This hypothesis ensures that

- $\delta(M) = dim_k\overline{R}/M = \#\ \mathbb{N} \setminus \Gamma(M)$
- there exists $k \in \{0, ..., \delta\}$ such that $c(M) = c_k$
- $\#\Gamma(M)_{<c_k} = c_k - \delta(M) = l_R(M/t^{c_k}\overline{R})$, where

  $\Gamma(M)_{<c_k} := \{j \in \Gamma(M) \mid j < c_k\}$

Denoting by $p := \#\ \{\ x \in [1, e] \cap \mathbb{N},\ \ x \notin \Gamma(M)\ \}$, then:

- $\delta(M) \leq \begin{cases} p\ s & \text{if } c_k = se \\ p\ (s+1) & \text{if } c_k = se + b, \quad 2 \leq b \leq e - 1 \end{cases}$

The *dualizing module* of R is: $\omega_R = \{\alpha \in k\{\{t\}\}dt\ \ s.t.\ res(f\alpha) = 0\ \forall f \in R\}$. By means of the isomorphism $k\{\{t\}\}dt \simeq k\{\{t\}\}$ which maps $dt \longmapsto 1$ we shall identify ω_R with a fractional ideal. Hence we can define $\nu(\alpha dt) := \nu(\alpha)$.
We fix as *canonical ideal* of R the ideal $\widetilde{\omega}_R := ut^c\omega_R$, where $u \in \overline{R}$ is a unit such that $R \subset \widetilde{\omega}_R \subset \overline{R}$. In virtue of the given characterization one can see that

$$c(\widetilde{\omega}_R) = c \qquad \text{and} \qquad \Gamma(\widetilde{\omega}_R) = \{j \in \mathbb{Z} \mid c - 1 - j \notin \Gamma\}$$

On the other hand in [7], Satz 5, Jäger shows that any R-module N such that $\Gamma(N) = \{j \in \mathbb{Z} \mid c - 1 - j \notin \Gamma\}$ is isomorphic to $\widetilde{\omega}_R$. Here we prove a slightly more general statement.

Lemma 1.1 *Let N be a finitely generated torsion free R-module of rank 1 containing $t^c\overline{R}$.*

i) If $\Gamma(N) \subset \{j \in \mathbb{Z} \mid c-1-j \notin \Gamma\}$, then there exists a unit $u \in \overline{R}$ such that $uN \subset \widetilde{\omega}_R$.

ii) If $\Gamma(N) = \{j \in \mathbb{Z} \mid c-1-j \notin \Gamma\}$, then $uN = \widetilde{\omega}_R$.

Proof.
i) Fix a basis $\{\overline{n_{i_1}}, ..., \overline{n_{i_s}}\}$ of the k-vector space $N/t^c\overline{R}$ such that $\nu(n_{i_1}) < ... < \nu(n_{i_s})$; so, if $j \in \Gamma(N)_{<c}$, then $j \in \{\nu(n_{i_1}), ..., \nu(n_{i_s})\}$.
For all $i = 0, ..., c-2$ define $n_i = n_{i_j}$ if $i = \nu(n_{i_j})$ for some $1 \leq j \leq s$, and $n_i = 0$ otherwise; hence we can write $n_i = t^i u_i$ where u_i is a unit of $\overline{R}$ in the first case and $u_i = 0$ otherwise.
Step 1. We show that there exists a unit $u = 1 + b_1 t + b_2 t^2 + ... + b_{c-1}t^{c-1}, b_i \in k$, such that $Res(un_i/t^c) = 0$ for all i. Imposing that the residue of each element $un_i/t^c = (u/t^c)t^i u_i = (1/t^c + b_1/t^{c-1} + ... + b_{c-1}/t)(u_{i_0}t^i + u_{i_1}t^{i+1} + ...)$, $u_{i_h} \in k$, $u_{i_0} \neq 0$ if $i = \nu(n_{i_j})$, is equal to 0 for all $i = 0, ..., c-2$, we get the following linear system (reduced if we cut the equations $0 = 0$):

$$\begin{cases} u_{0_{c-1}} + u_{0_{c-2}}b_1 + ... + u_{0_0}b_{c-1} & = & 0 \\ u_{1_{c-2}} + u_{1_{c-3}}b_1 + ... + u_{1_0}b_{c-2} & = & 0 \\ .... & & \\ .... & & \\ u_{(c-2)_1} + u_{(c-2)_0}b_1 & = & 0. \end{cases}$$

Since the number of equations is $\leq c-1$ we find a solution $(b_1, ..., b_{c-1})$.
Step 2. We show that $Res(un_i r/t^c) = 0$ for all $r \in R$.
Suppose there exists an element $r \in R$ such that $Res(un_i r/t^c) \neq 0$, i.e., $un_i r/t^c = a_\alpha/t^\alpha + ... + a_1/t + a_0 + a$, where $a_i \in k$, $\alpha \geq 1$, $a \in \overline{R}$ and $a_1 \neq 0$: then $-\alpha \in \Gamma(ut^{-c}N)$, hence $-\alpha + c \in \{\nu(n_{i_1}), ..., \nu(n_{i_k})\}$. Assume e.g. $-\alpha + c = \nu(n_{i_1})$ so that $n_{i_1}u/t^c = h_\alpha/t^\alpha + ...$ By subtracting we obtain the element $(un_i r/t^c) - (a_\alpha n_{i_1} u/h_\alpha t^c)$ of valuation $\geq -\alpha + 1$. Proceeding in this manner, we can go on subtracting at each step terms of residue zero, until we find an element of $ut^{-c}N$ with valuation -1. This implies that $c - 1 \in \Gamma(N)$, which is impossible by hypothesis.
ii) From part i) we have: $uN \subset \widetilde{\omega}_R \subset \overline{R}$. The thesis follows immediately since $\dim(\widetilde{\omega}_R/uN) = \#\ \Gamma(\widetilde{\omega}_R) \setminus \Gamma(N) = 0$.

Lemma 1.1 points out the correspondence between $\{j \in \mathbb{Z} \mid c-1-j \notin \Gamma\}$ and the isomorphism class of the canonical module. It becomes natural to go on in this way considering for any integer c_k such that $c_k - 1 \notin \Gamma$ the numerical sets

$$J_k := \{j \in \mathbb{Z} \mid c_k - 1 - j \notin \Gamma\}$$

$$I_k := \{j \in J_k \mid 0 \leq j < c_k\}$$

These sets will be very useful in the next proposition.

Note that $\#I_k = k$. In fact the map $j \longmapsto c_k - j$ is a bijection between the sets I_k and $\{c_1, ..., c_k\}$.

Proposition 1.2 *Let R and Γ be as above. Let $R_k := R + t^{c_k}\overline{R}$, where c_k, $k = 0, ..., \delta$, are the positive integers such that $c_k - 1 \notin \Gamma$.*

i) $J_k - c_k = \Gamma(\omega_R)$ for all $k = 0, ..., \delta$.

ii) For any R-module $M \subset \overline{R}$ such that $c(M) = c_k$

a) $\Gamma(M)_{<c_k} \subset I_k$

b) there exists a unit $u \in \overline{R}$ such that $ut^{c-c_k}M \subset \widetilde{\omega}_R$.

iii) For all $k = 0, ..., \delta$ the R-modules M such that $\Gamma(M) = \Gamma(R_k)$ constitue a unique isomorphism class $\mathcal{R}_k$.

iv) For all $k = 0, ..., \delta$ the R-modules M such that $c(M) = c_k$ and $\Gamma(M)_{<c_k} = I_k$ constitue a unique isomorphism class $\mathcal{M}_k$ which is the class of the canonical ideal $\widetilde{\omega}_{R_k}$ of R_k.

Proof.
i) By definition $\Gamma(\omega_R) = \Gamma(\widetilde{\omega}_R) - c = \{j \in \mathbb{Z} \mid -1 - j \notin \Gamma\}$. Now $j \in J_k - c_k \Longleftrightarrow j + c_k \in J_k \Longleftrightarrow -1 - j \notin \Gamma \Longleftrightarrow j \in \Gamma(\omega_R)$.
ii) a) $j \in \Gamma(M) \Longrightarrow j \geq 0$ since $M \subset \overline{R}$. Suppose $j \notin J_k \Longrightarrow c_k - 1 - j \in \Gamma \Longrightarrow c_k - 1 \in \Gamma(M)$, a contradiction.
ii) b) Let $N := t^{c-c_k}M$. We claim that $\Gamma(N) \subset J_\delta$. In fact $j \in \Gamma(N) \Longrightarrow j = c - c_k + x,\ x \in \Gamma(M) \Longrightarrow c - 1 - j = -1 + c_k - x \notin \Gamma$, otherwise $c_k - 1 \in \Gamma(M)$, which is impossible. Then apply i) of Lemma 1.1 to the R-module N.
iii) It suffices to note that any M having $c(M) = c_k$ is also an R_k-module.
iv) The hypothesis $\Gamma(M)_{<c_k} = I_k$ implies that $\Gamma(M) = \{j \in \mathbb{Z} \mid c(R_k) - 1 - j \notin \Gamma(R_k)\}$; hence it suffices to apply ii) of Lemma 1.1, regarding M as R_k-module.

Assertion ii a) of Prop.1.2 says that any R-module M such that $c(M) = c_k$ and $R \subset M \subset \overline{R}$ verifies the inequality

$$c_k - \delta(M) \leq k$$

Next prop.1.5 will show that equality holds if and only if M is the canonical module of the overring R_k. There exists a better bound for the difference $c_k - \delta(M)$; to see this we introduce a new invariant strictly connected with the *Cohen-Macaulay type* $r_R(M)$ of M.
Recall that:

$$r_R(M) := \dim_k (M : \mathfrak{m}_R)/M = \#\Gamma(M : \mathfrak{m}_R) \setminus \Gamma(M)$$

Notation Given a semigroup $H \subset \mathbb{N}$ such that $\delta(H) := \# \ \mathbb{N} \setminus H < \infty$ and $H + \Gamma(M) \subset \Gamma(M)$ denote by

$$A_H(M) := \{x \in \mathbb{Z} \mid x + (H \setminus \{0\}) \subset \Gamma(M),\, x \notin \Gamma(M),\, x \neq c_k - 1\}$$

$$\alpha_H(M) := \# A_H(M)$$

$$\alpha(M) \quad := \# A_\Gamma(M) \quad \text{for simplicity when} \quad H = \Gamma$$

Moreover given two numerical sets $H,\ K,$ denote

$$H - K := \{x \in \mathbb{Z} \mid x + H \subset K\}$$

$$H^+ := H \cap \mathbb{N} \quad \text{and} \quad H^- := H \setminus H^+$$

The defined invariants of M regarded as an R_k-module can be obtained considering only the positive elements of the above sets:

Proposition 1.3 *Let* $c(M) = c_k$.

i) $r_k(M) := r_{R_k}(M) = \# \Gamma(M : \mathfrak{m}_R)^+ \setminus \Gamma(M)$

ii) $\alpha_k(M) := \alpha_{\Gamma(R_k)}(M) = \# A_\Gamma(M)^+$

Example
These inequalities may be strict:
let $R = k[\![t^8, t^{22}, t^{26}, t^{28}, t^{35}]\!]$ and $M = R + t^2 R + (t^{12} + t^{13})R + t^{20} R + t^{28} \overline{R}$. Here $c = 57$ and $c(M) = 28 = c_{22}$. An easy calculation gives $A_\Gamma(M) = \{-6, 4, 13, 23, 25\}$, while $\Gamma(M : \mathfrak{m}_R) \setminus \Gamma(M) = \{-6, 13, 23, 25, 27\}$, because no element of value 4 belongs to $M : \mathfrak{m}_R$. Hence $r_k(M) = 4,\ r_R(M) = 5,$ $\alpha_k(M) + 1 = 5$ and $\alpha(M) + 1 = 6$.

Remark

• In the case $H = \Gamma$ we have $A_\Gamma(M) = (\Gamma(M) - \Gamma(\mathfrak{m}_R)) \setminus (\Gamma(M) \cup \{c_k - 1\})$. The inclusion $\Gamma(M : \mathfrak{m}_R) \subset \Gamma(M) - \Gamma(\mathfrak{m}_R)$ says that $r_R(M) \leq \alpha(M) + 1$. Hence:

$$\begin{array}{ccc} \alpha_k(M) + 1 & \leq & \alpha(M) + 1 \\ | \!\bigvee & & | \!\bigvee \\ r_k(M) & \leq & r_R(M) \end{array}$$

• Notice that $x < c_k - c \implies x \notin \Gamma(M) - \Gamma(\mathfrak{m}_R)$. In fact, $x < c_k - c$ and $x \in \Gamma(M) - \Gamma(\mathfrak{m}_R)$ would imply $c_k - 1 - x \in \Gamma(\mathfrak{m}_R)$ and $x + (c_k - 1 - x) \in \Gamma(M)$, which is impossible. As a consequence of this simple observation we get

$$\alpha(M) - \alpha_k(M) \leq c - c_k$$

• Calling e the multiplicity of R, since two elements of $A_\Gamma(M) \cup \{c_k - 1\}$ have by definition distinct residues mod. e, it follows that $\alpha(M) + 1 \leq e$.

• Let $P := \{x \notin \Gamma(M),\ 0 < x \leq e-1\}$ and let $p := \# P$. Then

$$\alpha_k(M) + 1 \leq p$$

In fact let $m = he + b \in A_\Gamma(M)^+ \cup \{c_k - 1\}$, with $h \geq 0,\ 1 \leq b \leq e-1$; since $m \notin \Gamma(M) \Longrightarrow b \in P$, we have an injective map $A_\Gamma(M)^+ \cup \{c_k - 1\} \hookrightarrow P$.

• If $c_k \leq e$ then $\alpha_k(M) + 1 = r_k(M) = \delta(M)$.
In fact $x \geq 0,\ x \notin \Gamma(M) \implies t^x \in M : \mathfrak{m}_R$.

Proposition 1.4 *Let* $c(M) = c_k$ *and let* $H \subset \mathbb{N}$ *be a semigroup such that* $0 \in H$, $\delta(H) := \#(\mathbb{N} \setminus H) < \infty$ *and* $H + \Gamma(M) \subset \Gamma(M)$. *The following inequality holds:*

$$c_k - \delta(M) + \alpha_H(M) \leq \delta(H)$$

In particular

i) for $H = \Gamma(R_k)$ $\quad \alpha_k(M) \leq k - (c_k - \delta(M))$

ii) for $H = \Gamma$ $\quad \alpha(M) \leq \delta - (c_k - \delta(M))$

Proof. Write $\mathbb{N} \setminus H = \Delta \cup \Delta_H$, where $\Delta := \{j = c_k - 1 - i \ |i \in \Gamma(M)_{<c_k}\}$ and $\Delta_H := \{j \notin H \mid j = c_k - 1 - i > 0 \text{ and } i \notin \Gamma(M)\}$ (disjoint union) and note that $\#\Delta = \#\Gamma(M)_{<c_k} = c_k - \delta(M)$. Moreover there is an injective map $\psi : A_H(M) \longrightarrow \Delta_H$ defined by $\psi(x) = c_k - 1 - x$.

• If $M = R$, then $\alpha(R) = \alpha_\delta(R)$ is called the "asymmetry" of Γ.
It is well known (see [8]) that R is a Gorenstein ring if and only if the value semigroup Γ is symmetric, i.e., $\alpha(R) = 0$. The same holds for modules:

Proposition 1.5 *Let* $c(M) = c_k$. *The following facts are equivalent:*

a) $c_k = \delta(M) + k$

b) $M \simeq \widetilde{\omega}_{R_k}$

c) $r_k(M) = 1$

d) $\alpha_k(M) = 0$.

Proof. We first prove the equivalences assuming $k = \delta$, then the claim follows regarding M as an R_k-module.
a) $\Longrightarrow$ *b)* By hypothesis $\Gamma(M)_{<c} = I_\delta \Longrightarrow \Gamma(M) = \Gamma(\widetilde{\omega}_R) \Longrightarrow M \simeq \widetilde{\omega}_R$.
b) $\Longleftrightarrow$ *c)* is [5], Korollar 6.12.
b) $\Longrightarrow$ *d)* Suppose $\alpha(M) \neq 0 \Longrightarrow$ there exists $x \in A_\Gamma(\widetilde{\omega}_R) \Longrightarrow c - 1 - x \in \Gamma \setminus \{0\}$ and $x + (c - 1 - x) \in \Gamma(\widetilde{\omega}_R)$, which is impossible.
d) $\Longrightarrow$ *a)* $\alpha(M) = 0$ implies $r_R(M) = 1$, i.e. $M \simeq \widetilde{\omega}_R$. So by duality: $\delta(M) = \dim_k(\overline{R}/t^c\omega_R) = \dim_k(R : t^cR)/(\omega_R \colon \overline{R}) = \dim_k(R/t^c\overline{R}) = c - \delta$.

Example
Let $R = k[\![t^5, t^8, t^{19}]\!]$ and $M = R + tR + t^3\overline{R}$. Then $c_k = 3$, $k = 2$, $\Gamma(M) = J_2 \cap \mathbb{N}$ i.e. $M = \widetilde{\omega}_{R_2}$, where $R_2 = k[\![t^3, t^4, t^5]\!]$. Here $A_\Gamma(M) = \{-5, -4, -2, -1\}$ hence $\alpha_k(M) = r_k(M) - 1 = 0$, $\alpha(M) = 4$ and $r_R(M) = 5$.

Remark For $M = R$ Prop.1.4 and Prop. 1.5 give the well known facts:

$c + \alpha(R) \leq 2\delta$ (See e.g. [2], 2.13)

$c = 2\delta$ if and only if R is Gorenstein

2

A moduli space for finitely generated torsion free R-modules of rank 1 is constructed by G.M. Greuel and G. Pfister in the paper [4]. The starting point is the remark that for any isomorphism class of R-modules it is possible to choose a representative M verifying $I := t^{2\delta}\overline{R} \subset M \subset \overline{R}$ and $dim_k \overline{R}/M = \delta$.
So it becomes natural to associate M with one point of the Grassmannian $Gr(\overline{R}/I,\ \delta)$ of δ-codimensional subspaces of $\overline{R}/I$.

The *reduced module variety* $\mathcal{M}(R) := Gr(\overline{R}/I,\ \delta)^{(R/I)^*}_{red}$ is then defined as the fixed point scheme with its reduced structure under the action, via multiplication, of the algebraic group $(R/I)^*$ on this Grassmannian.
It is easy to see that isomorphism classes of R-modules correspond to orbits under the action on $\mathcal{M}(R)$ of $(\overline{R}/I)^*/k^*$, which is canonically isomorphic to the Jordan group; the dimension of the orbit for a given M is $\delta(M{:}M)$.

Modules with the same value set may have orbits of different dimension.
To see this it suffices to consider the example after proposition 1.3, where
$R = k[\![t^8, t^{22}, t^{26}, t^{28}, t^{35}]\!]$ and $M = R + t^2R + (t^{12} + t^{13})R + t^{20}R + t^{28}\overline{R}$.
If $M_0 := R + t^2R + t^{12}R + t^{21}R + t^{28}\overline{R}$, then $\Gamma(M) = \Gamma(M_0)$, hence $\alpha(M) = \alpha(M_0)$ and $\alpha_k(M) = \alpha_k(M_0)$, but $r_R(M_0) = 6 > r_R(M) = 5$ and $\delta(M_0 :M_0) =$
$19 < \delta(M{:}M) = 20$. The reason is that $t^{10} \in M_0 :M_0$, but no element of valuation 10 belongs to $M{:}M$.

Using the same notations and assumptions as in the previous section, for any R-module M having conductor c_k we shall try to relate the orbit dimension $\delta(M{:}M)$ to the invariants of M defined above.

Among the semigroups considered in prop.1.4 the maximal one and its
δ-invariant in [4] are denoted by:

$\Gamma^0(M) := \{n \in \mathbb{N} \mid n + \Gamma(M) \subset \Gamma(M)\}$

$\delta^0(M) := \#\mathbb{N} \setminus \Gamma^0(M)$

Namely $\dfrac{c_k}{2} \leq \delta^0(M)$; this well-known inequality follows also from prop 1.4 applied to the semigroup ring S having $\Gamma^0(M)$ as value set, since $c(S) = c_k$. It is useful to note that there exist two natural bounds for the orbit dimension:

$$\delta^0(M) \leq \delta(M : M) \leq k$$

These inequalities hold by the inclusions $\Gamma(M{:}M) \subset \Gamma^0(M)$ and $M{:}M \supset R_k$; moreover the second one implies that if an R-module M has a δ-dimensional orbit then its conductor is c. Observe that:

• For all $k = 0, ..., \delta$ there exists at least one k-dimensional orbit in $\mathcal{M}(R)$, of course the class of isomorphism of R_k, by iii) of prop.1.2.

• It can happen that the lower bound δ^0 is not achieved by any orbit: let $R = k[\![t^8, t^{12} + t^{13}, t^{22}, t^{24}, \rightarrow]\!]$ and let $\Delta = \{\ 2k\}_{k=0,...,11} \cup [24, \infty)$ be a Γ-set. For any R-module M such that $R \subset M \subset \overline{R}$ and $\Gamma(M) = \Delta$ the orbit dimension is greater than δ^0. In fact $10 \in \Gamma^0(M)$, but no element of valuation 10 belongs to $M : M$. This is a simple verification.
Let $x = (1 + u_1t + u_2t^2 + ...)t^{10} \in M{:}M$. From $x(t^{12} + t^{13}) \in M$ we deduce $u_1 = -1$. Let now $y = (1+a_1t+a_2t^2+...)t^2 \in M$. Then $yt^8 = t^{10}+a_1t^{11}+a_2t^{12}+ ... \in M$. Since $xy = t^{12} + (u_1 + a_1)t^{13} + ... \in M$, it follows that $a_1 = 1 - u_1 = 2$. But with $u_1 = -1$ and $a_1 = 2$ the relation $x - yt^8 \in M$ is impossible.

This cannot happen if Γ is "monomial" (see section 3), because in this case for any Γ-set Δ the minimal value $\delta^0(\Delta)$ is achieved by the monomial module $M = k[\![\ t^i, i \in \Delta\ ]\!]$.

As a further application of prop. 1.4 we get

Proposition 2.1 *Let $c(M) = c_k$. Then:*

i) $c_k - \delta(M) + \ \alpha_{\Gamma^0(M)}(M) \quad \leq \delta^0(M)$

ii) $c_k - \delta(M) + \alpha_{\Gamma(M\ :\ M)}(M) \leq \delta(M{:}M)$

At this point we note that for all $k = 0, ..., \delta$ the R-module $M = \widetilde{\omega}_{R_k}$ has an orbit of minimal dimension $c_k - \delta(M)$; however, fixed a Γ-set $\Gamma(M)$, many orbits may have minimal dimension $c_k - \delta(M)$: let, for instance, $R = k[\![t^3, t^7]\!]$ and $\Gamma(M) = \{\ 0, 3, 4, 6 \rightarrow\}$; all the overrings $M = R + (1 + \lambda t)t^4R$, $\lambda \neq 0$ verify the equality $\delta(M{:}M) = c_k - \delta(M) = 3$ and for different values of λ the corresponding M are not isomorphic. About this problem we have

Proposition 2.2 *Let $c(M) = c_k$. The following facts are equivalent:*

i) $\delta(M{:}M) = c_k - \delta(M)$

ii) $M \simeq \widetilde{\omega}_{M:M}$

iii) $\alpha_{\Gamma(M:M)}(M) = 0$

In this case $\delta^0(M) = \delta(M{:}M)$ *and* $\alpha_k(M) + 1 = r_k(M)$.

Proof. Denote $S := M{:}M$. The equivalence of conditions is nothing else that prop.1.5 regarding M as an S-module and observing that $c(S) = c(M)$. It remains to prove that in this case $\Gamma(M) - \Gamma(\mathfrak{m}_R) \subset \Gamma(M : \mathfrak{m}_R)^+$. Let $x \in \Gamma(M) - \Gamma(\mathfrak{m}_R)$. Then $N := t^x \mathfrak{m}_R S + t^{c_k}\overline{R}$ is an S-module such that $\Gamma(N) \subset \Gamma(M) = \Gamma(\widetilde{\omega}_S)$. By lemma 1.1, i) $uN \subset \widetilde{\omega}_S = u'M$, u, u' units in $\overline{R}$, hence $ut^x\mathfrak{m}_R \subset u'M \Longrightarrow x \in \Gamma(M : \mathfrak{m}_R)^+$.

The key lemma in the sequel is:

Lemma 2.3 *Let* $c(M) = c_k$ *and let* $\mathfrak{a}, \mathfrak{b}$ *be fractional ideals such that* $\mathfrak{b} \subset \mathfrak{a} \subset \overline{R}$ *and* $M : \mathfrak{b} \subset \overline{R}$. *Then*

$$l_R(M{:}\mathfrak{b}/M{:}\mathfrak{a}) \leq l_R(\mathfrak{a}/\mathfrak{b})\ \ r_k(M)$$

Proof. Given a maximal chain of R-modules $\mathfrak{b} = M_n \subset ... \subset M_1 \subset M_0 = \mathfrak{a}$ we have $l_R(M : \mathfrak{b}/M : \mathfrak{a}) = \sum_{i=0}^{n-1} l_R(M{:}M_{i+1}/M{:}M_i)$. Since M_i/M_{i+1} is simple, for all $i = 0, ..., n-1$, there exists $x_i \in M_i$ such that $M_i = M_{i+1} + x_iR$. By Nakayama's Lemma $\mathfrak{m}_R M_i \subset M_{i+1}$, i.e., $Ann_R(M_i/M_{i+1}) = \mathfrak{m}_R$. Then as in Hilfssatz 4 of [7] we get an injective map $\overline{\phi}_i : M{:}M_{i+1}/M{:}M_i \longrightarrow M{:}\mathfrak{m}_R/M$ induced by $\phi_i : M{:}M_{i+1} \longrightarrow M{:}\mathfrak{m}_R$, $\phi_i(z) = zx_i$. Our further assumptions $\mathfrak{a} \subset \overline{R}$ and $M{:}\mathfrak{b} \subset \overline{R}$ assure that $Im\ \overline{\phi}_i \subset M :_{\overline{R}} \mathfrak{m}_R/M$; hence $l_R(M{:}M_{i+1}/M{:}M_i) \leq r_k(M)$ for all $i = 0, ..., n-1$.

Proposition 2.4 *Let* $c(M) = c_k$. *We have the following diagram:*

$$\begin{array}{ccc} \delta(M{:}M) & \leq & [c_k - \delta(M) - (k - \delta(M{:}M))]\ \ r_k(M) \\ |\!\wedge & & |\!\wedge \\ k & \leq & (c_k - \delta(M))\ \ r_k(M) \end{array}$$

Moreover

i) $k = (c_k - \delta(M))\ \ r_k(M) \quad \Longrightarrow \quad M{:}M = R_k$

ii) $\delta(M{:}M) = k = c_k - \delta(M) \quad \Longleftrightarrow \quad M \simeq \widetilde{\omega}_{R_k}$

Proof.
We first prove that $\delta(M{:}M) \leq (c_k - \delta(M) - (k - \delta(M{:}M)))\ \ r_k(M)$.
Given the inclusions $t^{c_k}\overline{R} \subset R_k \subset M{:}M \subset M$, it is possible to complete to a maximal chain of lenght $n := l_R(M/t^{c_k}\overline{R})$:

$$M_n := t^{c_k}\overline{R} \subset ... \subset M_j := R_k \subset ... \subset M_p := M{:}M \subset ... \subset M_0 := M$$

and obtain:

$$M{:}M_0 \subset ... \subset M{:}M_j \subset ... \subset M{:}M_p \subset ... \subset M{:}M_n$$

Since $M{:}M_n = M{:}t^{c_k}\overline{R} = \overline{R}$, $M{:}M_0 = M{:}M$ and $M{:}M_i = M{:}M_{i+1} = M$ for

$i = j, ..., p-1$, it suffices to apply the above lemma setting $\mathfrak{b} = M_{i+1}$ and $\mathfrak{a} = M_i$ to get
$\delta(M{:}M) = l_R(\overline{R}/M{:}M) = \sum_{i=0}^{n-1} l_R(M{:}M_{i+1}/M{:}M_i) \leq [n-(j-p)]r_k(M)$. Now $n = l_R(M/t^{c_k}\overline{R})$ and $j-p = l_R(M{:}M/R_k) = k - \delta(M{:}M)$, so the thesis follows.
The remaining assertions are easy consequences of the inequality:

$$k \leq (c_k - \delta(M))\ r_k(M) - (k - \delta(M{:}M))(r_k(M) - 1)$$

which follows from the preceding one. In fact:
$k - (k - \delta(M{:}M)) = \delta(M{:}M) \leq (c_k - \delta(M))\ r_k(M) - (k - \delta(M{:}M))\ r_k(M)$.

Corollary 2.5 *Consider M as a $(M{:}M)$-module. Then:*

$$\delta(M{:}M) \leq [c(M) - \delta(M)]\ r_{M:M}(M)$$

Proof. It is the first row of 2.4, applied with $R = M{:}M$, because $c(M) = c(M{:}M) = c_h$, where $h = \delta(M{:}M)$.

Remark

- For $M = R$ Cor. 2.5 becomes Theorem 1 of [1]:

$$(\cdot) \qquad \delta \leq (c - \delta)r(R)$$

Also in the not-Gorenstein case, this bound may be smaller than the natural one:

$$\delta \leq \begin{cases} c - \left[\frac{c}{e}\right] - 1 & \text{if } c = he + b, \quad 2 \leq b \leq e-1 \\ c - \left[\frac{c}{e}\right] & \text{if } c = he \end{cases}$$

For instance, let $R = k[\![t^9, t^{10}, t^{16}, t^{20}]\!]$. Here $c = 34$, $\delta = 20$, $A_\Gamma = \{31\}$, hence $\alpha + 1 = r(R) = 2$. Then $(c-\delta)r(R) = 28 < c - \left[\frac{c}{e}\right] - 1 = 30$.

- In the quoted paper [1] the authors give the following definitions:

  R is *of maximal length* if $\delta = (c - \delta)r(R)$

  R is *of almost maximal length* if $\delta = (c - \delta)r(R) - 1$.

We shall use this terminology in the next theorem.

- For an overring S with $c(S) = c_k$ prop.2.4 gives $\delta(S) \leq (c_k - k)r_k(S)$ which may be sharper than the preceding $(\cdot)$:
let $R = k[\![t^4, t^9]\!]$ and $S = R + t^6R + t^8\overline{R}$; here $c_k = 8$, $k = 6$, $\delta(S) = 5$, $r_k(S) = r(S) = 3$, hence $(c_k - k)r_k(S) = 6 < (c_k - \delta(S))r(S) = 9$.

- The inequality in 2.5 may be better than the first one of prop. 2.4:
let $R = k + t^8\overline{R}$, $S = k[\![t^4, t^5, t^{11}]\!]$ and $M = R + tR + t^4R + t^5R + t^6R = \widetilde{\omega}_S$.
Here $c_k = c(M) = 8$, $k = 7$, $\delta(M) = 3$, $r_S(M) = 1$, $r_k(M) = 3$. Therefore

$\delta(M{:}M) = [c(M)-\delta(M)]\ \ r_S(M) = 5 < \ \ [c_k-\delta(M)-(k-\delta(M{:}M))]r_k(M) = 9.$

• The number $[c_k - \delta(M) - (k - \delta(M{:}M))]r_k(M)$ may be smaller than k: let $R = k[\![t^4, t^9]\!]$ and $M = R + t^2R + t^8\overline{R}$. It is very easy to calculate $c_k = 8,\ k = 6,\ \delta(M{:}M) = \delta(M) = 4,\ r_k(M) = \alpha_k(M) + 1 = 2$, so $\delta(M{:}M) = 4 = [c_k - \delta(M) - (k - \delta(M{:}M))]r_k(M) < k = 6 < (c_k - \delta(M))\ \ r_k(M) = 8.$

• We note that in general the inequality $k < (c_k - \delta(M))\ \ r_k(M)$ holds also for the modules having an orbit of maximal dimension k:
let $R = k[\![t^4, t^9]\!]$ and $M = R_7$. Here $c_k = 11,\ k = 7,\ \ r_k(M) = 3.$

Looking at the preceding diagram we now ask whether some orbit of dimension $(c_k - \delta(M))\ r_k(M)$ corresponds to a given Γ-set $\Gamma(M)$. This can happen only in two particular situations: $\Gamma(M) = \Gamma(R_k)$, for very special k, or $\Gamma(M) = \Gamma(\widetilde{\omega}_{R_k})$ for all k. In both cases all the R-modules having $\Gamma(M)$ as value set constitue a unique orbit. We can characterize these possibilities by using the notion of rings of maximal length.

Theorem 2.6 *Let $R \subset M \subset \overline{R}$ and let $c_k = c(M)$.*

i) $\delta(M{:}M) = [c_k - \delta(M) - (k - \delta(M{:}M))]\ \ r_k(M) \Longleftrightarrow M$ is an overring and $\delta(M) = (c_k - k)\ \ r_k(M)$ or $M \simeq \widetilde{\omega}_{R_k}$

ii) $\delta(M{:}M) = (c_k - \delta(M))\ \ r_k(M) \Longleftrightarrow M$ is a ring of maximal lenght and CM-type $r_M(M) = r_k(M)$ or $M \simeq \widetilde{\omega}_{R_k}$

iii) $\delta(M{:}M) = (c_k - \delta(M))\ \ r_k(M) - 1 \Longleftrightarrow M$ is a ring of almost maximal lenght and CM-type $r_M(M) = r_k(M)$ or $r_k(M) = l_R(M/M{:}M) + 1 = 2$, $\delta(M) = 2h,\ c_k = 3h + 1,\ h \geq 1$.

The structure of the rings R and M verifying ii) or iii) will be explicitely described in the next prop. 2.7.

Proof of i) and ii) $\Longleftarrow$

If $M \simeq \widetilde{\omega}_{R_k}$, then by prop.1.5 $\alpha_k(M) + 1 = r_k(M) = 1$ and $\delta(M{:}M) = c_k - \delta(M)$. So the thesis holds. Otherwise $r_k(M) \neq 1$, then i) is trivial and ii) holds by definition.

Proof of iii) $\Longleftarrow$

If $r_k(M) = l_R(M/M{:}M) + 1 = 2$, then $\delta(M{:}M) = l_R(\overline{R}/M{:}M) = \delta(M) + 1 = 2h + 1$ and $(c_k - \delta(M))r_k(M) - 1 = (h + 1) \cdot 2 - 1 = 2h + 1.$
If M is a ring of almost maximal lenght and CM-type $r_k(M)$, then equality holds by definition.

Proof of i) $\Longrightarrow$

As in the proof of prop. 2.4 complete the inclusions $t^{c_k}\overline{R} \subset R_k \subset M{:}M \subset M$, to a maximal chain of lenght $n := l_R(M/t^{c_k}\overline{R})$:

$M_n := t^{c_k}\overline{R} \subset ... \subset M_j := R_k \subset ... \subset M_p := M{:}M \subset ... \subset M_0 := M$

and obtain:

$M{:}M = M{:}M_0 \subset ... \subset M{:}M_p = M \subset ... \subset M{:}M_j \subset ... \subset M{:}M_n = \overline{R}$

Recall that $M{:}M_i = M$ for $i = j, ..., p$. Using our hypothesis we get $\delta(M{:}M) = \sum_{i=0}^{n-1} l_R(M{:}M_{i+1}/M{:}M_i) = [n - (j-p)]r_k(M)$ where $n = l_R(M/t^{c_k}\overline{R})$ and $j - p = l_R(M{:}M/R_k) = k - \delta(M{:}M) \Longrightarrow$ by lemma 2.3 $l_R(M{:}M_{i+1}/M{:}M_i) = r_k(M)$ for all $i = 0, ..., p-1, j, ..., n-1$. We now compare $l_R(M/M{:}M)$ in both the above chains: it results $p = l_R(M/M{:}M) = \sum_{i=0}^{p-1} l_R(M{:}M_{i+1}/M{:}M_i) = p \cdot r_k(M)$, so either $r_k(M) = 1$ (equivalently $M \simeq \widetilde{\omega}_{R_k}$ by prop.1.5) or $p = 0$,i.e., $M = M{:}M$ is a overring.

<u>Proof of ii) $\Longrightarrow$</u>

The diagram of prop.2.4 and the hypothesis give $k = \delta(M{:}M)$, hence equality i) holds, then $M \simeq \widetilde{\omega}_{R_k}$, or $M = M{:}M$ is an overring of R such that $\delta(M) = k \Longrightarrow M = R_k$. Moreover: $k = l_R(\overline{R}/M) \leq l_R(M/t^{c_k}\overline{R})r_M(M) \leq l_R(M/t^{c_k}\overline{R})r_k(M)$ (see Theorem 1 of [1]). So our hypothesis implies that $r_M(M) = r_k(M)$ and M is a ring of maximal lenght by definition.

<u>Proof of iii) $\Longrightarrow$</u>

Using the same argument as in the first part of i) we conclude that the assumption $\delta(M{:}M) = l_R(\overline{R}/M{:}M) = \sum_{i=0}^{n-1} l_R(M{:}M_{i+1}/M{:}M_i) = n \cdot r_k(M) - 1$ implies $l_R(M{:}M_{i+1}/M{:}M_i) = r_k(M) - 1$ for one $i \in \{0, ..., n-1\}$ and $l_R(M{:}M_{j+1}/M{:}M_j) = r_k(M)$ for $j \neq i$.

Case 1) $j \leq i \leq n-1$.

As above we find $p = p \cdot r_k(M)$. Since $r_k(M) = 1$ contradicts the hypothesis, we deduce $p = 0$, i.e., $M = M{:}M$. Moreover

$\delta(M{:}M) \leq (c_k - \delta(M))\ r_k(M) - (k - \delta(M{:}M))\ r_k(M) = \delta(M{:}M) + 1 - (k - \delta(M{:}M))\ r_k(M) \Longleftrightarrow (k - \delta(M{:}M))\ r_k(M) \leq 1 \Longleftrightarrow k - \delta(M{:}M) = 0 \Longleftrightarrow M = M{:}M = R_k$.

It remains to prove that $r_M(M) = r_k(M)$. Applying Theorem 1 of [1] to the ring M and using the equality of the hypothesis, we have $l_R(\overline{R}/M) = t \cdot r_k(M) - 1 \leq t \cdot r \leq t \cdot r_k(M)$ where $t := l_R(M/t^{c_k}\overline{R})$ and $r := r_M(M)$. Hence there are only two cases: $t \cdot r_k(M) - 1 = t \cdot r$ or $t \cdot r = t \cdot r_k(M)$. We show that the first one is impossible. In fact, $t \cdot r_k(M) - 1 = t \cdot r \Longrightarrow t(r_k(M) - r) = 1 \Longrightarrow t = 1$, i.e., $t^{c_k}\overline{R} = \mathfrak{m}_M$ and $r = r_k(M) - 1$. Now $r_k(M) - 1 = r_M(M) = l_R(M{:}\mathfrak{m}_M/M) < l_R(M{:}_{\overline{R}}\mathfrak{m}_R/M) = r_k(M)$ means that there exists $x \in \overline{R}$ such that $x\mathfrak{m}_R \subset M$ and $x\mathfrak{m}_M \not\subset M$, which is absurd because $\mathfrak{m}_M\overline{R} \subset M$. So we conclude $r = r_k(M)$, hence M is a ring of almost maximal lenght by definition.

Case 2) $0 \leq i \leq p-1$.

Considering the right part of the chain as in i) we find $p = (p-1)r_k(M) + r_k(M) - 1 \Longrightarrow p(r_k(M) - 1) = 1 \Longrightarrow p = r_k(M) - 1 = 1$. Then $l_R(\overline{R}/M{:}M) = \delta(M) + 1 = (c_k - \delta(M))2 - 1 \Longrightarrow \delta(M) = 2h, c_k = 3h + 1$.

The proof of theorem 2.6 is now complete.

In the quoted paper [1] the authors state some general structure theorems for rings of maximal and almost maximal length. It is now natural to ask what the further condition $r(M) = r_k(M)$ means for such rings. In both cases we are able to give an answer.

Proposition 2.7 *Let M be an overring of R, $M \subset \overline{R}$, and $c(M) = c_k$. Let e_M be the multiplicity of M and e the multiplicity of R.*
Put $r(M) := r_M(M)$.

i) If $c_k > e_M$ and $r(M) = r_k(M)$, then $e_M = e$.

ii) M has maximal length and $r(M) = r_k(M)$ if and only if
either $M = R_k$ and R_k is Gorenstein
or $M = R_k$ and $\Gamma(R_k) = \{0, e_M, 2e_M, 3e_M, ..., pe_M, pe_M + 1 \rightarrow\}$.

iii) M has almost maximal length and $r(M) = r_k(M)$ if and only if $M = R_k$ and:
if $r(M) \geq 3$ then $\Gamma(R_k) = \{0, e_M, 2e_M, ..., (p-1)e_M, pe_M - 1 \rightarrow\}$;
if $r(M) = 2$ then either $\Gamma(R_k) = \{0, 3, 6, ..., 3h - 1 \rightarrow\}$
or $\Gamma(R_k) = \{0, 4, 7, 8, 11 \rightarrow\}$ or $\Gamma(R_k) = \{0, 4, 5, 8 \rightarrow\}$.

Proof.
i) Of course $e \geq e_M$, since $R \subset M$. Suppose now $e > e_M$. The assumption $r(M) = r_k(M)$ means $\Gamma(M : \mathfrak{m}_M) = \Gamma(M : \mathfrak{m}_R)^+$. But $c_k - 1 - e_M \notin \Gamma(M : \mathfrak{m}_M)$ and $c_k - 1 - e_M \in \Gamma(M : \mathfrak{m}_R)^+$, a contradiction.
The "only if" parts of ii) and iii) are easy computations.
ii) By Theorem 4 of [1] and subsequent Corollary there are two possibilities: either M is Gorenstein or $\mathfrak{m}_M = (x, \mathcal{C})$ where $\mathcal{C} = (x^p)\overline{R}$, $p \geq 1$, is the conductor ideal of M. In the first case the hypothesis $r(M) = 1$ implies $r_k(M) = 1$, i.e., $M \simeq \widetilde{\omega}_{R_k}$. Hence $M = M : M = R_k$ and $c_k = 2k$. In the case M not Gorenstein, by the quoted result $\Gamma(M) = \{0, e_M, 2e_M, ..., pe_M, pe_M + 1 \rightarrow\}$, $c_k = pe_M$, $\delta(M) = p(e_M - 1)$, $r(M) = e_M - 1$.
We claim that $M = R_k$, where $k = p(e_M - 1)$.
If $c_k = e_M$, then $\Gamma(M) = \{0, e_M \rightarrow\} \implies M = R_k$, with $k = e_M - 1$. If $c_k > e_M$, then by i) $e = e_M \implies \mathfrak{m}_M = (x, t^{c_k}\overline{R})$ where $\nu(x) = e \implies \Gamma(M) = \Gamma(R_k) \implies M = R_k$ and $k = p(e_M - 1)$.
iii) By [1] 4, Prop., either $r_M(M) \geq 3$ and $r(M) = e_M - 1$ or $r(M) = 2$ and $e_M = 3$ or $e_M = 4$.
If $r_M(M) \geq 3$, then by [1], 4, Corollary 1, $\mathfrak{m}_M = (x, \mathcal{C})$ and $l_M(\mathcal{C}/x^p\overline{R}) = 1$ for some transversal $x \in \mathfrak{m}_M$, $p \geq 2$. Moreover by [1], 4, Corollary 2, we know that $\Gamma(M) = \{0, e_M, 2e_M, ..., (p-1)e_M, pe_M - 1 \rightarrow\}$, $c_k = pe_M - 1$, $\delta(M) = p(e_M - 1) - 1$ and $r(M) = e_M - 1$.

If $c_k = e_M$, then $\Gamma(M) = \{0, e_M \rightarrow\}$ which is absurd because M would be of maximal length. Hence $e_M < c_k$ holds and, by i), $e = e_M$. So $\mathfrak{m}_M = (x, t^{c_k}\overline{R})$ where $\nu(x) = e \Longrightarrow \Gamma(M) = \Gamma(R_k) \Longrightarrow M = R_k$ where $k = p(e_M - 1) - 1$.
If $r(M) = 2$, then by [1], 4, Proposition, $e_M = 3$ or $e_M = 4$.

Case 1) Let $r(M) = 2$, $r_k(M) = 2$, $e_M = 3$.
By hypothesis $\delta(M) = 2(c_k - \delta(M)) - 1 \Longrightarrow 3\delta(M) = 2c_k - 1 \Longrightarrow \delta(M) = 2h - 1$ and $c_k = 3h - 1$, $h \geq 2$. Since $\#\{x \in \mathbb{N} \mid x < 3h - 1 \text{ and } x \notin 3\mathbb{N}\} = 2h - 1 = \delta(M)$, it must be $\Gamma(M) = \{0, 3, 6, ..., 3h - 1 \rightarrow\}$. On the other hand part i) says that $3 \in \Gamma$, hence $\Gamma_{<c_k} = \Gamma(M)_{<c_k} \Longrightarrow M = R_k$.

Case 2) Let $r(M) = 2$, $r_k(M) = 2$, $e_M = 4$.
As in case 1) we get $\delta(M) = 2h - 1$ and $c_k = 3h - 1$, $h \geq 2$. There are the following possibilities:

a)	$h = 4k$	$\delta(M) = 8k - 1$	$c_k = 12k - 1$	$t(M) = k$	$y \geq 8k - 1$
b)	$h = 4k + 1$	$\delta(M) = 8k + 1$	$c_k = 12k + 2$	$t(M) = k$	$y \geq 8k + 2$
c)	$h = 4k + 2$	$\delta(M) = 8k + 3$	$c_k = 12k + 5$		
d)	$h = 4k + 3$	$\delta(M) = 8k + 5$	$c_k = 12k + 8$	$t(M) = k + 1$	$y \geq 8k + 4$

Here $t(M) := \#\{j \in \mathbb{N} \mid j \in \Gamma(M)_{<c_k},\ j \notin 4\mathbb{N}\}$ and y denotes any element such that $y \in \Gamma(M)$ and $y \notin 4\mathbb{N}$.
First of all observe that case c) cannot happen because the condition $c_k = 12k+5$ contradicts the fact that $c_k - 1 \notin \Gamma(M)$. Easily one computes $t(M)$. In fact $t(M) = c_k - \delta(M) - s$ where $s := \#\{j \in 4\mathbb{N},\ j < c_k\}$. Now we note that if $y \in \Gamma(M) \setminus 4\mathbb{N}$, then $y + 4p \in \Gamma(M) \setminus 4\mathbb{N}$ for all $p \in \mathbb{N}$. Comparing the number of elements of the set $\{j \in \mathbb{N} \mid j = y + 4p,\ p \in \mathbb{N} \textit{ and } j < c_k\}$ with $t(M)$, we obtain the inequalities of the last column in the table.
Let y_1, y_2, y_3 be the minimal numbers of $\Gamma(M)$ having resp. residue $1, 2, 3$ $mod.4$ and let $m_0, m_1, m_2, m_3 \in M$ of valuation resp. $4, y_1, y_2, y_3$. The elements $m_1/m_0, m_2/m_0, m_3/m_0 \notin M$; if they belong to $M : \mathfrak{m}_M$, then $r(M) = 3$, against our hypothesis. The last column of the table says that $\nu(m_j m_i/m_0) = y_j + y_i - 4 \geq c_k, \forall i, j \neq 0$ (hence $r(M) = 3$) if $k \geq 2$ in case a), $k \geq 1$ in b) and $k \geq 1$ in d). Therefore only two situations are possible for M: case a) with $k = 1$ and case d) with $k = 0$.
In the first one necessarily $\Gamma(M) = \{0, 4, 7, 8, 11 \rightarrow\}$; since we need to have $r_k(M) = 2$, an easy computation shows that $\Gamma_{<c_k} = \Gamma(M)_{<c_k}$, hence $M = R_7$. In the second one necessarily $\Gamma(M) = \{0, 4, 5, 8 \rightarrow\}$ and the same computation as above shows that $M = R_5$.

Remark
In the cases ii) and iii) of theorem 2.6 the CM-type of M is maximal and $\alpha_k(M) + 1 = r_k(M) = e_M - 1$.

Examples

In these examples the equalities of theorem 2.6 are verified.

Let R and M be the semigroup rings with the following value sets respectively:

- Case of item i):

$\Gamma(R) = \{0, 5, 8, 10, 13 \rightarrow\}$ and $\Gamma(M) = \{0, 5, 8, 9, 10, 13 \rightarrow\}$, $c_k = 13$, $k = 9$, $\delta(M) = 8$, $r_k(M) = 2$.

- Case of maximal lenght, $r(M) = r_k(M)$ and Gorenstein:

$M = R_k$, $\Gamma(R_k) = \{0, 4, 5, 6, 8 \rightarrow\}$, $c_k = 8$, $k = 4$, $\alpha_k(M) + 1 = r_k(M) = r(M) = 1$

- Case of maximal lenght, $r(M) = r_k(M)$ and not Gorenstein:

let R be any ring with $e(R) = e$ and let $M = R_k$ with $k = e - 1$, so $\Gamma(M) = \{0, e \rightarrow\}$, $c(M) = e$, $r_k(M) = r(M) = e - 1$.

- Case of almost maximal lenght and $r(M) = r_k(M)$:

$M = R_k$, $\Gamma(R_k) = \{0, 4, 8, 11 \rightarrow\}$, $c_k = 11$, $k = 8$, $\alpha_k(M) + 1 = r_k(M) = r(M) = 3$.

3

Monomial semigroups. In [9] a semigroup Γ in $\mathbb{N}$ is called *monomial* if $0 \in \Gamma$, $\#(\mathbb{N} \setminus \Gamma) < \infty$ and each reduced and irreducible curve singularity with semigroup Γ has the local ring isomorphic to $k[\![t^{k_1}, ..., t^{k_m}]\!]$, where $\{k_1, ..., k_m\}$ are generators of Γ. In the quoted paper it is proved that a semigroup is monomial if and only if it is one of the following

(i) $\Gamma_{e,s,b} := \{ie, i = 0, 1, ..., s\} \cup [se + b, \infty)$ with $1 \leq b < e$, $s \geq 1$

(ii) $\Gamma_{e,r} := \{0\} \cup [e, e + r - 1] \cup [e + r + 1, \infty)$ with $2 \leq r \leq e - 1$

(iii) $\Gamma_e := \{0, e\} \cup [e + 2, 2e] \cup [2e + 2, \infty)$ with $e \geq 3$

In this last section we want to apply our preceding results on invariants of R-modules to the case $\Gamma(R)$ monomial. So we shall assume without loss of generality that $R = k[\![t^{k_1}, ..., t^{k_m}]\!] \subset M \subset \overline{R}$, $M \neq \overline{R}$.

We shall prove, in particular, that the C.M.-type of M depends only on the value set, since $r(M) = \alpha(M) + 1$ holds in each case.

Let, as usually, $c(M) = c_k$ and put $r(M) := r_R(M)$.

Proposition 3.1 *Let $\Gamma = \Gamma_{e,s,b}$ and let $P := \{x \notin \Gamma(M),\ 0 < x \leq e - 1\}$.*

i) $r(M) = \alpha(M) + 1$

ii) $r_k(M) = \alpha_k(M) + 1 = p$ *where* $p := \#P$

iii) *a) if* $c(M) = c$ *then* $r(M) = p$

b) if $c - e < c(M) < c$ *then*
$r(M) = c - c_k + \#\{x \notin \Gamma(M) \mid 1 \leq x < e - c + c_k\}$

c) if $c(M) \leq c - e$ *then* $r(M) = e$.

Proof. i) and ii) We first see that $\Gamma(M : \mathfrak{m}_R) = \Gamma(M) - \Gamma(\mathfrak{m}_R)$.
In fact $x \in \Gamma(M) - \Gamma(\mathfrak{m}_R) \implies ut^{x+e} \in M$, u a unit in $\overline{R}$ and $x \geq c_k - c$ by the second remark after prop. 1.3 $\implies ut^{x+ke} \in M$ for all $k \geq 1$ and $ut^{x+j} \in M$ for all $j \geq c \implies ut^x \mathfrak{m}_R \subset M$. To complete the proof we find a bijection between P and $A_\Gamma(M)^+ \cup \{c_k - 1\}$. For each $x \in P$ let $h \in \mathbb{N}$ be the minimal positive such that $x + (h+1)e \in \Gamma(M)$ and $x + he \notin \Gamma(M)$. Then φ: $x \longrightarrow x + he$ is the required map.
iii) a) follows from i) and ii).
iii) b) Observe that $A_\Gamma(M)^- = \{x \mid c_k - c \leq x < 0 \text{ and } x + e \in \Gamma(M)_{<e}\}$; then $x \longrightarrow x + e$ is a biunivocal correspondence between $A_\Gamma(M)^-$ and the set $Q := \{x \in \Gamma(M) \mid e - (c - c_k) \leq x \leq e - 1\}$. Using i) and ii) we get $r(M) = \alpha(M) + 1 = p + \#Q$, hence the thesis.
iii) c) Let $m_i \in \Gamma(M)$ be the first element of residue i mod e; it is straightforward to check that $m_i - e \in A_\Gamma(M)$.

Proposition 3.2 *Let* $\Gamma = \Gamma_{e,r}$.

i) If $c(M) = c = e + r + 1$ *then* $\alpha(M) = \delta(M) - r - 1$

ii) If $c(M) = c_k \leq e$ *then*

a) $\alpha_k(M) + 1 = \delta(M)$

b) $\alpha_k(M) + e - c_k \leq \alpha(M) \leq \alpha_k(M) + e - c_k + 1$

iii) $r_k(M) = \alpha_k(M) + 1$

iv) $r(M) = \alpha(M) + 1$

Proof. i) If $i \in \mathbb{N}$ is such that $0 < i \leq r$ then $i \notin \Gamma(M)$ and $i \notin A_\Gamma(M)$, hence $A_\Gamma(M) = \{ x \notin \Gamma(M) \mid r < x < e \}$.
ii) Assertion a) holds in general by the last remark after prop.1.3. To calculate $\alpha(M)$ observe that $-e \leq j < 0$, $j \in A_\Gamma(M) \implies c_k - 1 \neq j + e + h \in \Gamma(M)\ \forall\, h \in \mathbb{N} \setminus \{r\}$. Thus either $j \geq c_k - e$ or $j = c_k - 1 - r - e$, $c_k \geq r + 1$ and $\{ x \in [j + e, \infty),\ x \neq c_k - 1\} \subset \Gamma(M)$. So we can conclude that if $c_k \geq r + 1$ then $\alpha_k(M) + e - c_k \leq \alpha(M) \leq \alpha_k(M) + e - c_k + 1$, otherwise $\alpha_k(M) + e - c_k = \alpha(M)$.
iii) is a consequence of i). In fact, $x \geq 0$, $x \in \Gamma(M) - \Gamma(\mathfrak{m}_R) \implies t^x \mathfrak{m}_R \subset t^{c_k}\overline{R} \subset M$ since $x > r$ if $c(M) = c$.
iv) In the case $\alpha_k(M) + e - c_k = \alpha(M)$ it follows immediately since $t^j \in M : \mathfrak{m}_R$ for each $j \in [c_k - e, 0)$. In the other case $\alpha_k(M) + c_k - e + 1 = \alpha(M)$, we saw that $r \leq c_k - 1 \leq e - 1$ and $c_k - 1 - r - e \in A_\Gamma(M)$, so it is enough to prove that there exists a unit $u = 1 + a_1 t + a_2 t^2 + \ldots + a_r t^r \in \overline{R}$, $a_i \in k$, such that $ut^{c_k - 1 - r - e} \in M : \mathfrak{m}_R$, equivalently the following system in r equations:

$$\begin{cases} (1+a_1t+a_2t^2+...+a_rt^r)\ t^{c_k-1-r} & \in \quad M \\ (1+a_1t+...+a_{r-1}t^{r-1})\ t^{c_k-r} & \in \quad M \\ .... & \\ .... & \\ (1+a_1t+a_2t^2)\ t^{c_k-3} & \in \quad M \\ (1+a_1t)\ t^{c_k-2} & \in \quad M \end{cases}$$

admits a solution $(a_1,...,a_r)$. Since $c_k-1 \notin \Gamma(M)$ and by i) $c_k-2 \in \Gamma(M)$, using the last equation we get a_1, so step by step the triangular form of the system allows us to find the solution.

Proposition 3.3 *Let* $\Gamma = \Gamma_e$.

i) If $c(M) = c$ *then* $M \simeq R$ *and* $r(M) = 1$

ii) If $c(M) = e+2$ *then*

a) $\alpha_k(M) = \delta(M) - 2$

b) $\alpha_k(M) \leq \alpha(M) \leq \alpha_k(M) + 1$

iii) If $c(M) \leq e$ *then*

a) $\alpha_k(M) + 1 = \delta(M)$

b) $\alpha_k(M) + e - c_k \leq \alpha(M) \leq \alpha_k(M) + e - c_k + 1$

iv) $r_k(M) = \alpha_k(M) + 1$

v) $r(M) = \alpha(M) \quad +1$

Proof. i) The hypothesis $c(M) = 2e+2$ implies $\Gamma(M) = \Gamma$, hence the thesis.
ii) It is clear that $A_\Gamma(M)^+ = \{x \notin \Gamma(M)\} \setminus \{1, c_k - 1\}$. The only negative number which can belong to $A_\Gamma(M)$ is $-e$ and if $-e \in A_\Gamma(M)$ then $\Gamma(M) = \{0\} \cup [2,e] \cup [e+2, \infty)$.
iii) a) is always true. In order to compute the negative numbers which belong to $A_\Gamma(M)$, write $j = -e+h \in A_\Gamma(M)$ with $0 \leq h < e$. If $0 \leq h < c_k$ then the only possibility is $h = c_k - 2 \in \Gamma(M)$, whereas for each $c_k \leq h < e$ certainly $-e+h \in A_\Gamma(M)$.
iv) Since it is always true in the case $c(M) \leq e$ and $r_k(M) = r(M) = 1$ in the case $c(M) = c$, we suppose $c(M) = e+2$. In this case $x \in A_\Gamma(M)^+ \implies t^x \in M : \mathfrak{m}_R$.
v) In the case $c(M) = e+2$, if $-e \in A_\Gamma(M)$, then it is easy to check that there exists a unit $u \in \overline{R}$ such that $ut^{-e} \in M : \mathfrak{m}_R$, using the same argument as in the above proof, claim iv). In the case $c(M) \leq e$ for each $c_k \leq h < e$ certainly $t^{-e+h} \in M : \mathfrak{m}_R$, so the only thing to check is that there exists a unit $u \in \overline{R}$ such that $ut^{-e+c_k-2} \in M : \mathfrak{m}_R$. We conclude as above.

Finally, we want to state what the first equality of theorem 2.6

$$\text{(o)} \qquad \delta(M) = (c_k - k)\ r_k(M)$$

means in the "monomial case" for an overring M of R.
Suppose Γ is a *monomial* semigroup and $R = k[\![t^{k_1}, ..., t^{k_m}]\!]$ its semigroup ring.

We begin observing that in the case $\Gamma = \Gamma_{e,s,b}$ inequality of theorem 2.6: $\delta(M) \leq (c_k - k)\ r_k(M)$ becomes the natural one. More precisely if M is an overring of R with $c(M) = c_k = te + b'$, $t \leq s$, $0 \leq b' \leq e-1$, then we get

$$\delta(M) \leq \begin{cases} p\ t & \text{if } b' = 0 \\ p\ (t+1) & \text{if } b' > 1 \end{cases}$$

In fact, if $c_k = te$ then $c_k - k = t$ and if $c_k = te + b'$, $b' > 1$ then $c_k - k = t + 1$; moreover $r_k(M) = p$ by prop 3.1.

Proposition 3.4
Let M be an overring of R with $c(M) = c_k = te + b'$, $0 \leq b' \leq e-1$, then

- *i) If $c(M) \leq e$, then equality (o) is verified by any overring of R*
- *ii) Case $\Gamma = \Gamma_{e,s,b}$.*
  Equality (o) holds if and only if

  $t = 0$ *and M is any overring*

  $b' = 0,\ t = 1$ *and M is any overring*

  $b' = 0,\ t > 1$ *and $\Gamma(M) = \Gamma_{e',t',1}$ where e' divides e and $t' = te/e'$*
- *iii) Case $\Gamma = \Gamma_{e,r}$.*
  Equality (o) holds if and only if either $c(M) < c$ or $c(M) = c$ and M is the canonical module of R.
- *iv) Case $\Gamma = \Gamma_e$.*
  Equality (o) holds if either $c(M) = c$ or $c(M) \leq e$ or $c(M) = e+2$ and $\Gamma(M) = \{0\} \cup [2, e] \cup [e+2, \infty)$, i. e., $\delta(M) = 2$.

Proof. i) In this case $\delta(M) = \alpha_k(M) + 1$ and $c_k - k = 1$.
ii) Suppose $b' = 0$: the above equality becomes $\delta(M) = tp$, trivial if $t = 1$, hence assuming $t > 1$ we see that in each interval $[je, (j+1)e]$ the gaps of $\Gamma(M)$ are exactly p. Let e' be the first element of $\Gamma(M)$, $0 < e' \leq e$ and let $e = he' + r$, $r < e' \Longrightarrow (h+1)e' \in \Gamma(M)$ and $e < (h+1)e' < e + e'$ if $r > 0 \Longrightarrow (h+1)e'$ would have residue $< e'$. Hence $r = 0$ and iterating the same argument the thesis follows.
Suppose $b' > 1$, then equality (o) becomes $\delta(M) = (t+1)p$. Therefore in each interval $[je, je+b]$ the gaps of $\Gamma(M)$ are exactly p and $[je+b, (j+1)e] \subset \Gamma(M)$, this would imply reasoning as above that $e - 1/e$, absurd.
iii) In fact, $c(M) = c = e + r + 1$, $\delta = e$. Then $\delta(M) = r + \alpha + 1 = (r+1)(\alpha+1) \Longrightarrow \alpha = 0$, hence by prop.1.5 $M \simeq \widetilde{\omega}_R \simeq R$, i.e., $r = e - 1$.
iv) In fact, if $c(M) = c$, then $M \simeq R$ and R is Gorenstein. The remaining is a straightforward computation.

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